AUTOMORPHISMS OF $\Sigma_{n+1}^{-}$ INVARIANT TRILINEAR FORMS

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Abstract. Examination of automorphism groups of forms is undertaken by many authors. Sometimes the description of such groups is a difficult task. It turns out that a representation of a form as a sum of powers of linear forms may be very helpful, especially when this representation is unique. We show this in the case of $\Sigma_{n+1}^{-}$ invariant symmetric trilinear form $\Theta$ considered by Egawa and Suzuki.

Any $d-$linear symmetric form $\Theta : V^d \rightarrow K$ on the $K-$vector space $V$ determines the form (homogeneous polynomial) of degree $d$ defined by

$$f_\Theta(X_1, \ldots, X_n) := \Theta(\sum_{i=1}^{n} X_i e_1, \ldots, \sum_{i=1}^{n} X_i e_i),$$

where $(e_1, \ldots, e_n)$ is the basis of $V$. In the sequel suppose that $K$ is a field of characteristic 0. Then, by the polarization formula, the correspondence $\Theta \mapsto f_\Theta$ is bijective. An automorphism $\varphi$ of $\Theta$ is any automorphism $\varphi$ of $V$ such that

$$\Theta(\varphi(\alpha_1), \ldots, \varphi(\alpha_d)) = \Theta(\alpha_1, \ldots, \alpha_d), \text{ for } \alpha_1, \ldots, \alpha_d \in V.$$

The automorphisms of $\Theta$ form a group $\text{Aut}(\Theta)$. Of course, $\varphi \in \text{Aut}(\Theta)$ if and only if $\varphi \in \text{Aut}(f_\Theta)$, that is,

$$f_\Theta(\varphi(X_1, \ldots, X_n)) = f_\Theta(X_1, \ldots, X_n).$$

It means that $\text{Aut}(\Theta) = \text{Aut}(f_\Theta)$.

It is known that the space $F_{n,d}(K)$ of forms over $K$ of degree $d$ in $n$ variables is spanned by $d-$th powers of linear forms (see [5, Proposition 2.11]). Therefore any form $f \in F_{n,d}(K)$ over an algebraically closed field $K$ can be written in the following way

(1) $$f = l_1^d + \ldots + l_r^d, \text{ where } l_j^d = (\alpha_{1j} X_1 + \ldots + \alpha_{nj} X_n)^d, \quad \alpha_{ij} \in K.$$  

It is obvious that any automorphism of $V$ that permutes the summands in (1) belongs to $\text{Aut}(f)$. Thus, if we know the representation (1) of $f$, then we get certain information about automorphisms of $f$. It can happen that when $r$ is fixed the representation (1) of $f$ is unique (that is, the linear forms $l_1, \ldots, l_r$ are unique up to reordering and multiplying by $d-$th roots of unity). In such a case we can derive from (1) the complete information about $\text{Aut}(f)$. Uniqueness of the representation of forms as a sum of powers of linear forms was discussed by many authors in the previous century. However, usually generic forms were considered. Nice exposition of the results presents the book by A. Iarrobino and V. Kanev [4]. Information on unique representation of special forms one can find among others in [5], [1] and [2].

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The influence of a given representation (1) of $f$ on the structure of $\text{Aut}(f)$ will be examined in [6]. In this paper we show how it works in one special case.

Egawa and Suzuki [3] considered a trilinear form constructed in the following way. Let $\Sigma_{n+1}$ be the symmetric group on the set $\{0,1,\ldots,n\}$ for $n \geq 2$ and let $V = \langle e_1, \ldots, e_n \rangle$ be the natural $n$-dimensional irreducible $\Sigma_{n+1}$-module over the complex number field $\mathbb{C}$. That means that $(e_1, \ldots, e_n)$ is a basis of $V$ and $\Sigma_{n+1}$ acts on $\{e_0, e_1, \ldots, e_n\}$ in a standard way, where $e_0 = -(e_1 + \cdots + e_n)$. A $\Sigma_{n+1}$-invariant symmetric trilinear form $\Theta_n$ on $V$ was defined by

$$\Theta_n(e_j, e_j, e_j) = n(n-1), \quad 1 \leq j \leq n,$$

$$\Theta_n(e_j, e_j, e_k) = -(n-1), \quad 1 \leq j, k \leq n, \quad j \neq k,$$

$$\Theta_n(e_j, e_k, e_l) = 2, \quad 1 \leq j, k, l \leq n, \quad j \neq k \neq l \neq j.$$ 

They proved that if $\Theta$ is an arbitrary nonzero $\Sigma_{n+1}$-invariant symmetric trilinear form, then $\Theta = a\Theta_n$, $0 \neq a \in \mathbb{C}$. The main result of their paper was the following theorem.

**Theorem.** [3, Theorem 2] If $n = 2$ or $n \geq 4$, then $\text{Aut}(\Theta_n) \cong \mu_3 \times \Sigma_{n+1}$, where $\mu_3$ is the group of complex 3rd roots of unity.

The proof of this theorem was quite long. It took several pages, used a few lemmas and required separately the case of even and odd $n$. The aim of this paper is to show for $n \geq 4$ a very short proof using a representation of $f_{\Theta_n}$ as a sum of third powers of linear forms. Observe that

$$f_{\Theta_n}(X_1, \ldots, X_n) = \Theta_n\left(\sum_{i=1}^{n} X_i e_i, \sum_{i=1}^{n} X_i e_i, \sum_{i=1}^{n} X_i e_i\right) =$$

$$= n(n-1) \sum_{i=1}^{n} X_i^3 - 3(n-1) \sum_{i,j=1}^{n} X_i^2 X_j + 12 \sum_{i,j,k=1}^{n} X_i X_j X_k =$$

$$= -\frac{1}{n+1}((X_1 + X_2 + \ldots + X_n)^3 + (-nX_1 + X_2 + \ldots + X_n)^3 +$$

$$+ (X_1 - nX_2 + \ldots + X_n)^3 + \ldots + (X_1 + \ldots - nX_n)^3).$$

**Proof.** Suppose $n \geq 4$. Notice that if we apply the nonsingular linear substitution

$$X_i \mapsto -\frac{1}{\sqrt{n+1}}(X_1 + \ldots + X_{i-1} - nX_i + X_{i+1} + \ldots + X_n), \quad i = 1, \ldots, n$$

to the form $f_{\Theta_n}$, then we get the form

$$g_n(X_1, \ldots, X_n) := (-X_1 - \ldots - X_n)^3 + X_1^3 + \ldots + X_n^3.$$ 

Thus it suffices to consider $g_n$ instead of $f_{\Theta_n}$. It can be readily verified that $g_n$ is nondegenerate which means that $(0, \ldots, 0)$ is the only common zero of the partial derivatives $\partial^2 g_n / \partial X_i \partial X_j$, for $i, j = 1, \ldots, n$.

Now we shall show that $g_n$ is indecomposable, that is, $g_n$ can not be transformed by a nonsingular linear substitution to a form $g(X_1, \ldots, X_k) + h(X_{k+1}, \ldots, X_n)$,
for some forms $g \in F_{k,3}(K), h \in F_{n-k,3}(K)$ and $k \in \{1, \ldots, n-1\}$. Suppose that $g_n$ is decomposable and for some nonsingular linear substitution $\varphi$

$$f(X_1, \ldots, X_n) := g_n(\varphi(X_1, \ldots, X_n)) = g(X_1, \ldots, X_n) + h(X_{k+1}, \ldots, X_n), \ 1 \leq k \leq n-1.$$ 

Applying the hessian $H$ to the sides of the above equality we have

$$(2) \quad H_f(X_1, \ldots, X_n) = H_g(X_1, \ldots, X_k)H_h(X_{k+1}, \ldots, X_n).$$

Now taking into account that $g_n$ the above representation of $g_n$ as a sum of $n+1$ third powers of linear forms $l_0, l_1, \ldots, l_n$ is unique and every automorphism permutes the summands in this representation. Consider the group homomorphism

$$\Phi : \text{Aut}(g_n) \rightarrow \Sigma_{n+1}, \ \Phi(\varphi) = \sigma \iff l_0^3(\varphi(X_1, \ldots, X_n)) = l_0^3(\sigma(X_1, \ldots, X_n)).$$

By [2, Theorem 3.4] that says

"If a nondegenerate and indecomposable form of degree $d \geq 3$ in $n \geq 4$ variables is a sum of $n+1$ $d-$th powers of linear forms, then the linear forms are unique up to reordering and multiplying by $d-$th roots of unity, the above representation of $g_n$ as a sum of $n+1$ third powers of linear forms $l_0, l_1, \ldots, l_n$ is unique and every automorphism permutes the summands in this representation. Consider the group homomorphism

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Notice that cycles $(0, i)$ and $(1, \ldots, n)$ belong to im$\Phi$, so $\Phi$ is an epimorphism. Moreover, ker $\Phi = \mu_3 \text{id}_V$. In this way we get the exact sequence

$$0 \rightarrow \mu_3 \text{id}_V \rightarrow \text{Aut}(g_n) \rightarrow \Sigma_{n+1} \rightarrow 0$$

which splits. Since $\mu_3 \text{id}_V$ is contained in the center of $\text{Aut}(g_n)$ we have

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Now taking into account that $g_n$ is a reducible polynomial. However, this is not true, because by Eisenstein criterion, $H_{g_n}$ is an irreducible polynomial. Let

$$l_0(X_1, \ldots, X_n) = -(X_1 + \ldots + X_n), \ l_i(X_1, \ldots, X_n) = X_i, \ i = 1, \ldots, n.$$ By [2, Theorem 3.4] that says

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**Remark.** The proof presented above works over any field $K$ of characteristic 0 which contains a primitive third root of unity. We have not to worry about the coefficient $-\frac{1}{\sqrt[n+1]{3}}$ used in the proof, because at the beginnig we could have considered the form $-(n+1)f_{\Theta_n}$ instead of $f_{\Theta_n}$. In case $K$ lacks a primitive third root of unity we can easily reorganize the proof to get $\text{Aut}f_{\Theta_n} \cong \Sigma_{n+1}$. 

\[\square\]
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REFERENCES


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