10 Rational functions field of an affine algebraic variety.

Definition 10.1. *Let $V \subseteq k^n$ be an affine algebraic variety. The field of fractions of the coordinate ring $k[V]$ will be called the* ***field of rational functions*** *of V and denoted by $k(V)$, and its elements* ***rational functions*** *on V.*

Example 10.2.

- $V = \{(a_1, \ldots, a_n)\} \in k^n$, $k(V) \cong k$;
- $V = k^k$, $k(V) \cong k(x_1, \ldots, x_n)$.

Definition 10.3. *Let $V \subseteq k^n$ be an affine algebraic variety. A rational function $\varphi \in k(V)$ is **defined** at a point $(a_1, \ldots, a_n) \in V$ if $\varphi = \frac{f}{g}$, for some $f, g \in k[V]$ with $g(a_1, \ldots, a_n) \neq 0$. In this case we say that $\frac{f(a_1, \ldots, a_n)}{g(a_1, \ldots, a_n)} \in k$ is the **value** of φ at $(a_1, \ldots, a_n)$, and denote it by $\varphi(a_1, \ldots, a_n)$.*

Remark 10.4. Let $V \subseteq k^n$ be an affine algebraic variety, let $\varphi \in k(V)$ be defined at $(a_1, ..., a_n) \in V$. The value of φ at $(a_1, ..., a_n)$ is uniquely defined.

Example 10.5. Let $V = \mathcal{Z}(x^2 + y^2 - 1) \subseteq \mathbb{C}^2$. Then $\mathbb{C}(V) \cong \mathbb{C}(x, y)$ with $x^2 + y^2 = 1$. Let $\varphi = \frac{1 - y}{x} \in \mathbb{C}(V)$. Then φ is defined at $(0, 1) \in V$ and $\varphi(0, 1) = 0$, but φ is not defined at $(0, -1)$.

Remark 10.6. Let $V \subseteq k^n$ be an affine algebraic variety. Every element $\frac{f}{g} \in k(V)$, $f, g \in k[V]$, determines a rational function defined on some nonempty open subset $U \subseteq V$ with values in k.

Remark 10.7. Let $V \subseteq k^n$ be an affine algebraic variety. If the rational functions $\varphi_1, \varphi_2 \in k(V)$ have the same values on a certain nonempty open subset of $U \subseteq V$, then they are equal.

Theorem 10.8. *Let* $V \subseteq k^n$ *be an affine algebraic variety. If the rational function* $\frac{f}{g} \in k(V)$, f, $g \in k[V]$, *is defined at every point of* V, *then* $\frac{f}{g} = \frac{h}{1}$, *for some* $h \in k[V]$.

Definition 10.9. *Let $V \subseteq k^n$ be an affine algebraic set, let $V = V_1 \cup \ldots \cup V_m$ be the decomposition of V into affine algebraic varieties. The **k-algebra of rational functions** of V is defined to be*

$$k(V) = k(V_1) \oplus \ldots \oplus k(V_m)$$

*and its elements are called **rational functions** on V.*

Definition 10.10. *Let $V \subseteq k^n$ be an affine algebraic set. If a rational function $\varphi \in k(V)$ is defined at every point of an open subset $U \subseteq V$, then the restriction $\varphi \restriction_U$ wil be called a* ***regular*** *function on U.*

Example 10.11. Let $V = \mathcal{Z}(xy)$. Then $V = \mathcal{Z}(x) \cup \mathcal{Z}(y)$. Let $f = x(y+1)$. Then $f\restriction_{\mathcal{Z}(x)\setminus\{(0,0)\}} = 0$ and $f\restriction_{\mathcal{Z}(y)\setminus\{(0,0)\}} = 1$, $f \in k(V)$, f is regular on both $\mathcal{Z}(x)$ and $\mathcal{Z}(y)$, but not regular on V, as it is not defined on $(0,0)$.

Remark 10.12. Let $V \subseteq k^n$ be an affine algebraic set, let $f \in k(V)$. Then f is continuous on the set of points where it is defined.

Theorem 10.13. *Let $V \subseteq k^n$ be an affine algebraic variety, let $f \in k[V] \setminus \{0\}$, let*

$$k[V]_f = \left\{ h \in k(V) \,|\, h = \frac{g}{f}, m \in \mathbb{Z}, g \in k[V] \right\}$$

and

$$V_f = \{(a_1, \ldots, a_n) \in V \,|\, f(a_1, \ldots, a_n) \neq 0\}.$$

Then the k-algebra of regular functions on V_f is isomorphic to $k[V]_f$.

11 Rational maps of affine algebraic sets. Birational equivalence of affine algebraic sets.

Definition 11.1. *Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes. A **rational map** $f\colon V \to W$ is a map such that there exist $f_1, \ldots, f_m \in k(V)$ such that $f(a) = (f_1(a), \ldots, f_m(a))$, for all the points $a \in V$ where all the rational functions $f_1, \ldots, f_m \in k(V)$ are defined.*

Remark 11.2. Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes, let $f: V \to W$ be a rational map, $f = (f_1, \ldots, f_m)$ with $f_1, \ldots, f_m \in k(V)$. There exists an open set $\emptyset \neq U \subseteq V$ such that $f_1 \restriction_U, \ldots, f_m \restriction_U$ are regular on U. In other words, we can think of rational maps as defined on open subsets.

Remark 11.3. Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes, let $f_1, \ldots, f_m \in k(V)$. Then $f_1, \ldots, f_m$ define a rational map $f: V \to W$.

Remark 11.4. Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes, let $f\colon V \to W$ be a rational map and assume that $f(V)$ is dense in W. The map f defines a field embedding $f^*\colon k(W) \to k(V)$.

Definition 11.5. *Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes, let $f\colon V \to W$ be a rational map such that $f(V)$ is dense in W. The map f is a* ***birational equivalence*** *if there is a rational map $g\colon W \to V$ such that $g(W)$ is dense in V and*

$$f \circ g = 1_W \text{ and } g \circ f = 1_V.$$

In this case we say that V and W are ***birationally equivalent*** *or* ***birational****.*

Corollary 11.6. *Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes. Then V and W are birationally equivalent if and only if $k(V) \cong k(W)$.*

Example 11.7. Let $V = \mathcal{Z}(xy - 1)$ and $W = \mathcal{Z}(y)$, let $f\colon V \to W$ be given by $(x, y) \mapsto (x, 0)$. This is a birational equivalence, but not an isomorphism.

Example 11.8. Let $V = \mathcal{Z}(y)$ and $W = \mathcal{Z}(y^2 - x^3)$, let $f\colon V \to W$ be given by $(x, 0) \mapsto (x^2, x^3)$. This is a birational equivalence (the inverse map $g\colon W \to V$ being $(x, y) \mapsto \left(\frac{y}{x}, 0\right)$), but not an isomorphism.

Proposition 11.9. *Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes, let $f\colon V \to W$ be a birational equivalence. Then there exist open subsets $U \subseteq V$ and $U' \subseteq W$ which are isomorphic.*

Proposition 11.10. (Noether normalization lemma) *Let k be algebraically closed, and $k \subseteq K$ a finitely generated field extension. Then there exist elements $z_1, \ldots, z_{d+1} \in K$ with $K = k(z_1, \ldots, z_{d+1})$ such that $z_1, \ldots, z_d$ are algebraically independent over k, and z_{d+1} is separable over $k(z_1, \ldots, z_{d+1})$.*

Proposition 11.11. *Let $V \subseteq k^n$ be an affine variety. Then V is birationally equivalent to a hypersurface of some affine space k^m.*

A variety is called **rational** if it is birationally equivalent to k^n, for some n.