9 Coordinate ring of an affine algebraic set. Morphisms of affine algebraic sets. Category of affine algebraic sets.

9.1 Coordinate ring of an affine algebraic set.

Definition 9.1. *Let k be a field, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V. The ring $k[V] := k[x_1, \ldots, x_n]/\mathcal{I}(V)$ is called the* ***coordinate ring*** *of V.*

Remark 9.2. Let k be a field, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V. Let $f \in k[x_1, \ldots, x_n]$. The polynomial f defines a polynomial function $k^n \to k$. Let f_V be the restriction of f to the set V, $f_V = f \restriction_V$. Then $f_V = g_V$ if and only if $f + \mathcal{I}(V) = g + \mathcal{I}(V)$.

Remark 9.3. Let k be a field, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V. Let κ: $k[x_1, \ldots, x_n] \to k[V]$ be the canonical epimorphism, $\kappa(f) = \overline{f} := f + \mathcal{I}(V)$. Then $k[V]$ is a k-ring finitely generated over k by $\overline{x_1}, \ldots, \overline{x_2}$.

Remark 9.4. Let k be algebraically closed, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V. Then $k[V]$ has no nonzero nilpotents.

Theorem 9.5. *Let k be algebraically closed. Then a k-ring A is isomorphic to a coordinate ring of an affine algebraic set $V \subseteq k^n$ if and only if it is finitely generated over k and has no nonzero nilpotents.*

Example 9.6.

- $V = k^n$, $k[V] \cong k[x_1, \dots, x_n]$;
- $V = \emptyset$, $k[V] \cong 0$;
- $V = \{(a_1, \dots, a_n)\}$, $k[V] \cong k$.

Example 9.7. $V=\mathcal{Z}(f)$, $f\in k[x_1,\ldots,x_n]$ is square-free. $k[V]\cong k[x_1,\ldots,x_n]/(f)\cong k[\alpha_1,\ldots,\alpha_n]$ where $f(\alpha_1,\ldots,\alpha_n)=0$.

Example 9.8. $V = \mathcal{Z}(a_1x_1 + \ldots + a_nx_n - b)$, $k[V] \cong k[x_1, \ldots, x_{n-1}]$.

9.2 Category of affine algebraic sets.

Definition 9.9. *Let* $V \subseteq k^n$ *and* $W \subseteq k^m$ *be affine algebraic sets. A* ***morphism*** $f: V \to W$ *is a map such that there exist* $f_1, \ldots, f_m \in k[V]$ *such that* $f(a) = (f_1(a), \ldots, f_m(a))$, *for all* $a \in V$.

Remark 9.10. Let $V\subseteq k^n$ and $W\subseteq k^m$ be affine algebraic sets, let $f_1,\ldots,f_m\in k[V]$. Then $f=(f_1,\ldots,f_m)\colon V\to W$ is a morphism if and only if

$$g(f_1,\ldots,f_m)=0\in k[V]\quad\text{for all}\ g\in\mathcal{I}(W).$$

Example 9.11.

- Let $f\in k[V]$. Then $f\colon V\rightarrow k$ is a morphism.
- Let $f\colon k^n\rightarrow k^m$ be a linear map. Then f is a morphism.
- Let $f\colon\mathcal{Z}(xy-1)\rightarrow k$ be given by $f(x,y)=x$. Then f is a morphism.
- Let $f\colon k\rightarrow\mathcal{Z}(y^2-x^3)$ be given by $f(t)=(t^2,t^3)$. Then f is a morphism.

Example 9.12. One easily checks that:

- $\mathcal{Z}(y-x^k)\cong k$ via $f(x,y)=x$ and $g(t)=(t,t^k)$;
- $f\colon\mathcal{Z}(xy-1)\to k$ given by $f(x,y)=x$ is not an isomorphism;
- $f\colon k\to\mathcal{Z}(y^2-x^3)$ given by $f(t)=(t^2,t^3)$ is not an isomorphism, even though it is a bijection.

- We shall write $k-\mathcal{A}\mathrm{ff}$ for the category of affine algebraic sets over an algebraically closed field k with morphisms defined above.

Theorem 9.13. *Let k be algebraically closed and consider the categories $k-\mathcal{A}\mathrm{ff}$ and $k-\mathcal{A}\mathrm{lg}^0_{\mathrm{fg}}$. The assignment*

$$F(V) = k[V] \textit{ for an affine algebraic set } V \subseteq k^n$$

and

$$F(\varphi) = \varphi^* \ \textit{ for a morphism of affine algebraic sets } \ \varphi\colon V \to W,$$

where $\varphi^*\colon k[W] \to k[V]$ is given by the formula

$$\varphi^*(f) = g \circ \varphi$$

defines an equivalence of categories $k-\mathcal{A}\mathrm{ff}^{\mathrm{op}}$ and $k-\mathcal{A}\mathrm{lg}^0_{\mathrm{fg}}$.