

Definition and remark

Let \mathcal{C} be a category, let 0 be a zero object. Then for every pair of objects $C, D \in \text{Ob}(\mathcal{C})$ there exists exactly one morphism

$C \xrightarrow{0_{C,D}} D$ such that

$$f \circ 0_{C,D} = 0_{C,E} \text{ and } 0_{C,D} \circ g = 0_{B,D},$$

for all morphisms $h \in \text{Hom}(D, E)$ and $g \in \text{Hom}(B, C)$. The morphism $0_{C,D}$ is called the **zero morphism**.

Definition

Let \mathcal{C} be a category, let $C, D \in \text{Ob}(\mathcal{C})$, let $C \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} D$ be morphisms. An **equalizer** of the pair f, g is the pair (E, e) consisting of an object E and a morphism $E \xrightarrow{e} C$ such that

1. $f \circ e = g \circ e$;

2. if A is any object and $A \xrightarrow{h} C$ a morphism such that

$$f \circ h = g \circ h,$$

then there exists exactly one morphism $A \xrightarrow{\bar{h}} E$ such that

$$e \circ \bar{h} = h.$$

In other words the diagram

$$\begin{array}{ccccc} E & \xrightarrow{e} & C & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & D \\ \uparrow \bar{h} & & \nearrow h & & \\ A & & & & \end{array}$$

is commutative.

Definition:

Let \mathcal{C} be a category, let $C, D \in \text{Ob}(\mathcal{C})$, let $C \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} D$ be morphisms. A **coequalizer** of the pair f, g is the pair (Q, q) consisting of an object Q and a morphism $D \xrightarrow{q} Q$ such that

1. $q \circ f = q \circ g$;
2. if F is any object and $D \xrightarrow{k} F$ a morphism such that

$$k \circ f = k \circ g,$$

then there exists exactly one morphism $Q \xrightarrow{\bar{k}} F$ such that

$$\bar{k} \circ q = k.$$

In other words the diagram

$$\begin{array}{ccccc} C & \xrightarrow{f} & D & \xrightarrow{q} & Q \\ & \xrightarrow{g} & & & \downarrow \bar{k} \\ & & & \searrow k & F \end{array}$$

is commutative.

Remark

Let \mathcal{C} be a category, let $C, D \in \text{Ob}(\mathcal{C})$, let $C \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} D$ be morphisms, let (E, e) be the equalizer, and (Q, q) a coequalizer of f, g . Then:

1. e is a monic,
2. q is an epic.

Definition

Let \mathcal{C} be a category, let 0 be the zero object, let $C, D \in \text{Ob}(\mathcal{C})$, let $C \xrightarrow{f} D$ be a morphism. A **kernel** of f is the equalizer (E, e) of the pair $f, 0_{C,D}$. The equalizer (E, e) is then denoted by $(\text{Ker } f, \ker f)$.

A **cokernel** of f is the coequalizer (Q, q) of the pair $f, 0_{C,D}$. The coequalizer (Q, q) is then denoted by $(\text{Coker } f, \text{coker } f)$.

Corollary

Let \mathcal{C} be a category, let 0 be the zero object, let $C, D \in \text{Ob}(\mathcal{C})$, let $C \xrightarrow{f} D$ be a morphism. Then

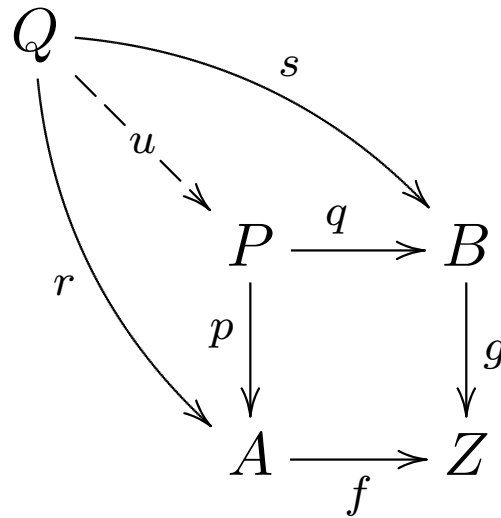
1. $\ker f$ is a monic and $f \circ \ker f = 0_{\text{Ker}f, D}$;
2. $\text{coker} f$ is an epic and $\text{coker} f \circ f = 0_{C, \text{Coker}f}$.

Definition:

Let \mathcal{C} be a category, let $A, B, Z \in \text{Ob}(\mathcal{C})$. Let $A \xrightarrow{f} Z$ and $B \xrightarrow{g} Z$ be morphisms. A **pullback** of the pair f, g is the triple (P, p, q) consisting of an object P and morphisms $P \xrightarrow{p} A$ and $P \xrightarrow{q} B$ such that:

1. $f \circ p = g \circ q$;
2. for every object Q and morphisms $Q \xrightarrow{r} A$ and $Q \xrightarrow{s} B$ such that $f \circ r = g \circ s$, there exists exactly one morphism $Q \xrightarrow{u} P$ such that $r = p \circ u$ and $s = q \circ u$;

in other words the diagram



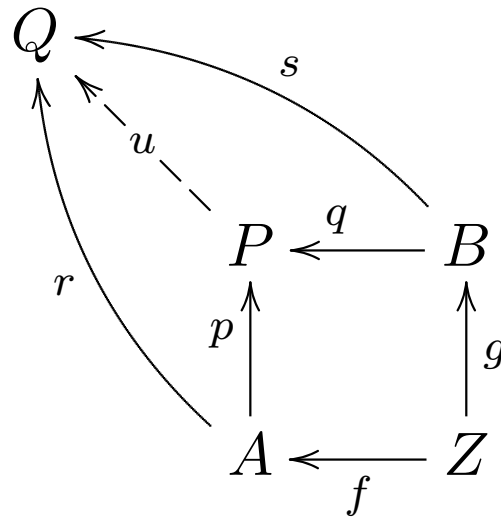
is commutative. The object P is denoted by $A \times_Z B$, the morphism p is called the **pulback of g along f** , and q is called the **pulback of f along g** .

Definition:

Let \mathcal{C} be a category, let $A, B, Z \in \text{Ob}(\mathcal{C})$. Let $Z \xrightarrow{f} A$ and $Z \xrightarrow{g} B$ be morphisms. A **pushout** of the pair f, g is the triple (P, p, q) consisting of an object P and morphisms $A \xrightarrow{p} P$ and $B \xrightarrow{q} P$ such that

1. $p \circ f = q \circ g$;
2. for every object Q and morphisms $A \xrightarrow{r} Q$ and $B \xrightarrow{s} Q$ such that $r \circ f = s \circ g$, there exists exactly one morphism $P \xrightarrow{u} Q$ such that $r = u \circ p$ and $s = u \circ q$;

in other words the diagram



is commutative. The object P is denoted by $A \cup_Z B$, the morphism p is called the **pushout of g along f** , and q is called the **pushout of f along g** .

Remark

Let \mathcal{C} be a category. Then binary products and equalizers in \mathcal{C} exist if and only if there exist pullbacks.

Remark

Let \mathcal{C} be a category. Then binary coproducts and coequalizers in \mathcal{C} exist if and only if there exist pushouts.

Definition

Let \mathcal{C} and \mathcal{D} be categories. A covariant functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a pair of maps (F_1, F_2) , $F_1 : Ob(\mathcal{C}) \rightarrow Ob(\mathcal{D})$, $F_2 : Ar(\mathcal{C}) \rightarrow Ar(\mathcal{D})$ such that

1. for $A \in Ob(\mathcal{C})$ and for $B = F_1(A) \in Ob(\mathcal{D})$

$$F_2(1_A) = 1_B;$$

2. for $A \xrightarrow{f} B$, $A, B \in Ob(\mathcal{C})$

$$F_1(A) \xrightarrow{F_2(f)} F_1(B);$$

3. for $B \xrightarrow{f} C$, $A \xrightarrow{g} B$, $A, B, C \in Ob(\mathcal{C})$:

$$F_2(f \circ g) = F_2(f) \circ F_1(g)$$

Definition

Let \mathcal{C} and \mathcal{D} be categories. A contravariant functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a pair of maps (F_1, F_2) , $F_1 : Ob(\mathcal{C}) \rightarrow Ob(\mathcal{D})$, $F_2 : Ar(\mathcal{C}) \rightarrow Ar(\mathcal{D})$ such that

1. for $A \in Ob(\mathcal{C})$ and for $B = F_1(A) \in Ob(\mathcal{D})$

$$F_2(1_A) = 1_B;$$

2. for $A \xrightarrow{f} B$, $A, B \in Ob(\mathcal{C})$

$$F_1(B) \xrightarrow{F_2(f)} F_1(A);$$

3. for $B \xrightarrow{f} C$, $A \xrightarrow{g} B$, $A, B, C \in Ob(\mathcal{C})$:

$$F_2(f \circ g) = F_2(g) \circ F_1(f)$$

Definition

Let \mathcal{C} and \mathcal{D} be categories, let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be covariant functors. A natural transformation λ from F to G is a class of morphisms $\lambda_A : F(A) \rightarrow G(A)$, $A \in \text{Ob}(\mathcal{C})$ such that for every $A \xrightarrow{\alpha} B$, $A, B \in \text{Ob}(\mathcal{C})$

$$G(\alpha) \circ \lambda_A = \lambda_B \circ F(\alpha).$$

In other words, the following diagram is commutative:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(\alpha)} & F(B) \\ \lambda_A \downarrow & & \downarrow \lambda_B \\ G(A) & \xrightarrow{G(\alpha)} & G(B) \end{array}$$

If λ_A are all isomorphisms, we say λ is a natural equivalence.

Definition

Let \mathcal{C} and \mathcal{D} be categories, let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be contravariant functors. A natural transformation α from F to G is a class of morphisms $\alpha_A : F(A) \rightarrow G(A)$, $A \in \text{Ob}(\mathcal{C})$ such that for every $A \xrightarrow{f} B$, $A, B \in \text{Ob}(\mathcal{C})$

$$F(f) \circ \alpha_A = \alpha_B \circ G(f).$$

In other words, the following diagram is commutative:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \uparrow \alpha_A & & \uparrow \alpha_B \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

If α_A are all isomorphisms, we say α is a natural equivalence.

Definition 7.14. Let \mathcal{C} and \mathcal{D} be categories. A covariant functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is **faithful** if for all objects $A, B \in \text{Ob}(\mathcal{C})$ the induced function

$$\text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(A), F(B))$$

is injective. If, moreover, it is surjective, then F shall be called **fully faithful**.

Proposition 7.15. *Let \mathcal{C} and \mathcal{D} be categories, let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a fully faithful functor. Then, for all objects $A, B \in \text{Ob}(\mathcal{C})$, $A \cong B$ if and only if $F(A) \cong F(B)$.*

Definition 7.16. Let \mathcal{C} and \mathcal{D} be categories. A covariant functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is called a **equivalence of categories** if it is fully faithful and essentially surjective, that is for every object $B \in \text{Ob}(\mathcal{D})$ there is an object $A \in \text{Ob}(\mathcal{C})$ such that $F(A) = B$.