

Categories.

Category \mathcal{C} consists of a class of **objects** $Ob(\mathcal{C})$, denoted by A, B, C, \dots and a class of **morphisms** (or **arrows**) $Ar(\mathcal{C})$ together with:

1. a class of pairwise disjoint arrows $Hom(A, B)$, one for each pair of objects $A, B \in Ob(\mathcal{C})$; an element f of $Hom(A, B)$ is called a **morphism** from A to B and denoted $A \xrightarrow{f} B$ or $f : A \rightarrow B$,
2. functions $Hom(B, C) \times Hom(A, B) \rightarrow Hom(A, C)$, for each triple $A, B, C \in Ob(\mathcal{C})$, called **composition of morphisms**; for morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ values of this function are denoted by $(g, f) \mapsto g \circ f$ and a morphism $A \xrightarrow{g \circ f} C$ called **composition** of morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$.

Moreover, the following axioms are satisfied:

Associativity: if $A \xrightarrow{f} B$, $B \xrightarrow{g} C$ and $C \xrightarrow{h} D$ are morphisms in \mathcal{C} , then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Identity: for each object $B \in \mathcal{C}$ there is a morphism $B \xrightarrow{1_B} B$ such that for each $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$:

$$1_B \circ f = f \text{ and } g \circ 1_B = g.$$

If the classes $Ob(\mathcal{C})$ are $Ar(\mathcal{C})$ sets, then the category \mathcal{C} is called **small**. If the classes $Hom(A, B)$, for each pair of sets $A, B \in Ob(\mathcal{C})$, are sets, then the category \mathcal{C} is called **locally small**.

An **isomorphism** is a morphism $A \xrightarrow{f} B$ such that there exists a morphism $B \xrightarrow{g} A$ such that

$$f \circ g = 1_B \text{ and } g \circ f = 1_A.$$

If there is an isomorphism $A \xrightarrow{f} B$, then the objects are **isomorphic**, denoted $A \cong B$.

An **automorphism** is an isomorphism $A \xrightarrow{f} A$. An **endomorphism** is a morphism $A \xrightarrow{f} A$.

Definition

Let \mathcal{C} be a category, let $\{A_i : i \in I\}$ be a family of objects in \mathcal{C} . A **product** of $\{A_i : i \in I\}$ is an object P together with a family of morphisms $\{P \xrightarrow{\pi_i} A_i : i \in I\}$ such that for each object B and a family of morphisms $\{B \xrightarrow{\phi_i} A_i : i \in I\}$ there is exactly one morphism $B \xrightarrow{\phi} P$ such that

$$\pi_i \circ \phi = \phi_i,$$

for $i \in I$. In other words, the following diagram commutes:

$$\begin{array}{ccc} B & \xrightarrow{\phi} & P \\ & \searrow \phi_i & \downarrow \pi_i \\ & & A_i \end{array}$$

A product P is denoted by $\prod_{i \in I} A_i$.

Definition

Let \mathcal{C} be a category, let $\{A_i : i \in I\}$ be a family of objects in \mathcal{C} . A **coproduct** of $\{A_i : i \in I\}$ is an object S together with a family of morphisms $\{A_i \xrightarrow{\iota_i} S : i \in I\}$ such that for each object B and a family of morphisms $\{A_i \xrightarrow{\psi_i} B : i \in I\}$ there is exactly one morphism $S \xrightarrow{\psi} B$ such that

$$\psi \circ \iota_i = \psi_i,$$

for $i \in I$. In other words, the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\psi} & B \\ \uparrow \iota_i & \nearrow \psi_i & \\ A_i & & \end{array}$$

A coproduct S is denoted by $\coprod_{i \in I} A_i$.

Definition

Let \mathcal{C} be a category. The category \mathcal{C} is called **concrete**, if there is a function $\sigma : \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\text{Set})$ such that

1. each morphism $A \xrightarrow{f} B$ is a function between sets $f : \sigma(A) \rightarrow \sigma(B)$;
2. the identity morphism $A \xrightarrow{1_A} A$ is the identity function $1_A : \sigma(A) \rightarrow \sigma(A)$;
3. composition of morphisms is composition of functions.

Definition

Let \mathcal{C} be a concrete category, let F be an object in \mathcal{C} , let X be a nonempty set, let $f : X \rightarrow F$ be a function between sets. The object F is called **free with basis** X if and only if for each object H and each function between sets $h : X \rightarrow H$ there is exactly one morphism $F \xrightarrow{\phi} H$ such that $\phi \circ f = h$.

Definition

Let \mathcal{C} be a category. An object I of \mathcal{C} is called an **initial object** (or **universal**), if for each object C of \mathcal{C} there is exactly one morphism $I \xrightarrow{i} C$.

An object T of \mathcal{C} is called a **terminal object** (or **couniversal**), if for each object C of \mathcal{C} there is exactly one morphism $C \xrightarrow{t} T$.

An object Z of \mathcal{C} is called a **zero object**, if it is both initial and terminal.

Definition

Let \mathcal{C} be a category, let $B, C \in \text{Ob}(\mathcal{C})$. A morphism $B \xrightarrow{\phi} C$ is called a **categorical monomorphism** (or **monic**), if for every object A and morphisms $A \xrightarrow[\psi_2]{\psi_1} B$:

$$\text{if } \phi \circ \psi_1 = \phi \circ \psi_2 \text{ then } \psi_1 = \psi_2.$$

A morphism $B \xrightarrow{\phi} C$ is called a **categorical epimorphism** (or **epic**), if for every object D and morphisms $C \xrightarrow[\psi_2]{\psi_1} D$:

$$\text{if } \psi_1 \circ \phi = \psi_2 \circ \phi \text{ then } \psi_1 = \psi_2.$$