7 Coordinate ring of an affine algebraic set. Morphisms of affine algebraic sets. Category of affine algebraic sets. Isomorphisms of affine algebraic sets.

1. Is the ring $\mathbb{Z}$ isomorphic to a coordinate ring of an affine algebraic set?

2. Let $V = \mathcal{Z}(f)$, where $f(x, y) = y - mx - b \in k[x, y]$. Show that $k[V] \cong k[x]$.

3. Let $V = \mathcal{Z}(f)$, where $f(x, y) = y - x^2 \in k[x, y]$. Show that $k[V] \cong k[x]$.

4. Let $V = \mathcal{Z}(xy - 1) \subseteq \mathbb{C}^2$. Show that $\mathbb{C}[V] \cong \mathbb{C}\big[x, \frac{1}{x}\big]$.

5. Show that coordinate rings of a circle and a hyperbola considered as affine algebraic sets in $\mathbb{C}^2$ are isomorphic.

6. Let $V = \mathcal{Z}(x^2 + y^2 - z^2) \subseteq \mathbb{C}^3$. Find $\mathbb{C}[V]$.

7. Let $V = \mathcal{Z}(x^2 + y^2 - z^2) \subseteq \mathbb{C}^3$, let $f = x^3 + 2xy^2 - 2xz^2 + x$ and $g = x - x^3$. Show that $f_V = g_V$.

8. Let $V = \mathcal{Z}(y^2 - x^3)$. Show that an element of $k[V]$ can be written uniquely in the form $p(x) + q(x)y$ with $p, q \in k[x]$.

9. Let $V = \{(t, t^2, t^3) | \, t \in k\}$. Show that V is an affine algebraic set and prove that $k[V] \cong k[x]$.

10. Prove that the hyperbola defined by $xy = 1$ and the line k^1 are not isomorphic as affine algebraic sets.

11. An isomorphism $f: V \to V$ of an affine algebraic set V is called an **automorphism**. Prove that all automorphisms of the line k^1 are of the form $f(x) = ax + b$ with $a \neq 0$.

12. Prove that the map $f(x, y) = (\alpha x, \beta y + P(x))$, where $\alpha, \beta \in k^\times$, $P(x) \in k[x]$, is an automorphism of k^2. Prove that all maps of that form a group. We shall call this group the group of **triangular** automorphisms.

13. Let V be an affine algebraic set consisting of two points. Prove that $k[V] \cong k \times k$.

14. Let $f: V \to W$ be a morphism of affine algebraic sets. The subset $\Gamma_f \subseteq V \times W$ consisting of all points of the form $(v, f(v))$ is called the **graph** of f. Prove that Γ_f is isomorphic to V.

15. Let $\operatorname{char} k = p$. Prove that the morphism $\psi: k^n \to k^n$ given by $\psi(x_1, ..., x_n) = (x_1^p, ..., x_n^p)$ is a bijection, but not an automorphism. We shall call it the **Frobenius automorphism.**

Homework: Problems 7, 8 and 9.