6 Categories.

1. Let $\mathcal{C}$ be a category, let $B, C, D \in \mathrm{Ob}(\mathcal{C})$, let $B \xrightarrow{\phi} C$ and $C \xrightarrow{\psi} D$ be morphisms. Show that:

   a. if ϕ and ψ are monics, then $\psi \circ \phi$ is a monic;

   b. if $\psi \circ \phi$is a monic, then ϕ is a monic;

   c. if ϕ and ψ are epics, then $\psi \circ \phi$ is an epic;

   d. if $\psi \circ \phi$ is an epic, then ψ is an epic;

   e. if ϕ is an isomorphism, then it is also both a monic and an epic.

2. Let $\mathcal{C}$ be a category, let $C \in Ob(\mathcal{C})$, let 0 be a zero object. Show that

   a. the uniquely defined morphism $0 \to C$ is a monic;

   b. the uniquely defined morphism $C \to 0$ is an epic.

3. Let $\mathcal{C}$ be a category, let $\{A_i: i \in I\}$ be a family of objects in $\mathcal{C}$. Let $\mathcal{D}$ be a category whose objects are pairs $(B, \{f_i: i \in I\})$, where $B \xrightarrow{f_i} A_i$ are morphisms in $\mathcal{C}$, $i \in I$, and morphisms from the class $\mathrm{Hom}((B, \{f_i: i \in I\}), (C, \{g_i: i \in I\}))$ are defined as morphisms $B \xrightarrow{h} C$ in $\mathcal{C}$ such that

   $$g_i \circ h = f_i.$$

   Show that the product $\prod_{i \in I} A_i$ exists in $\mathcal{C}$ if and only if $(\prod_{i \in I} A_i, \{\pi_i: i \in I\})$ is a terminal object in $\mathcal{D}$.

4. Let $\mathcal{C}$ be a category, let $\{A_i: i \in I\}$ be a family of objects in $\mathcal{C}$. Let $\mathcal{D}$ be a category whose objects are pairs $(B, \{f_i: i \in I\})$, where $A_i \xrightarrow{f_i} B$ are morphisms in $\mathcal{C}$, $i \in I$, and morphisms from the class $\mathrm{Hom}((B, \{f_i: i \in I\}), (C, \{g_i: i \in I\}))$ are defined as morphisms $B \xrightarrow{h} C$ in $\mathcal{C}$ such that

   $$h \circ f_i = g_i.$$

   Show that the coproduct $\coprod_{i \in I} A_i$ exists in $\mathcal{C}$ if and only if $(\coprod_{i \in I} A_i, \{\iota_i: i \in I\})$ is an initial object in $\mathcal{D}$.

5. Let $\mathcal{C}$ be a concrete category, let F be an object in $\mathcal{C}$, let X be a nonempty set, let $f: X \to F$ be a function between sets. Let $\mathcal{D}$ be a category whose objects are pairs (B, g), where g: $X \to B$ are functions between sets, and morphisms from the class $\mathrm{Hom}((B, g), (C, h))$ are defined as morphisms $B \xrightarrow{\phi} C$ in $\mathcal{C}$ such that

   $$\phi \circ g = h.$$

   Show that F is a free object with basis X in $\mathcal{C}$ if and only if (F, f) is an initial object in $\mathcal{D}$.

6. Let $\mathcal{C}$ be a category, let $C, D \in \mathrm{Ob}(\mathcal{C})$, let $C \underset{g}{\overset{f}{\rightrightarrows}} D$ be morphisms, let (E, e) be the equalizer, and (Q, q) a coequalizer of f, g. Show that:

   a. e is a monic,

   b. q is an epic.

7. Let $\mathcal{C}$ be a category, let 0 be the zero object, let $C, D \in \mathrm{Ob}(\mathcal{C})$, let $C \xrightarrow{f} D$ be a morhism. Show that:

   a. $\ker f$ is a monic and $f \circ \ker f = 0_{\mathrm{Ker}\, f, D}$;

   b. $\operatorname{coker} f$ is an epic and $\operatorname{coker} f \circ f = 0_{C, \mathrm{Coker}\, f}$.

8. Let $\mathcal{C}$ be a category. Show that binary coproducts and coequalizers in $\mathcal{C}$ exist of and only if there exist pushouts.

9. Let $\mathcal{C}$ be a category, let $C, D \in Ob(\mathcal{C})$, let $C \xrightarrow{f} D$ be a morphism. An **image** of f is the pair (I, m) consisting of an object I and a monic m such that:

   a. there exists a morphism $C \xrightarrow{e} I$ such that $f = m \circ e$;

   b. for any object I' with a morphism $C \rightarrow e'\, I'$ and any monic $I' \xrightarrow{m'} D$ such that $f = m' \circ e'$, there is a unique morphism $I \xrightarrow{v} I'$ such that $m = m' \circ v$.

   The object I is denoted by $\mathrm{Im}(f)$ and the morphism i by $\mathrm{im}(f)$.

   Show that in the category $\mathcal{S}et$ for a morphism $C \xrightarrow{f} D$ the pair consisting of the set-theoretical image of f and the inclusion map is the image.

10. Show that if the category $\mathcal{C}$ has all equalizers, then the morphism e in the factorization $f = m \circ e$ of the definition of image is an epic.

11. Let $\mathcal{C}$ be a category, let $C, D \in Ob(\mathcal{C})$, let $C \xrightarrow{f} D$ be a morphism. An **coimage** of f is the pair (J, c) consisting of an object J and a epic $C \xrightarrow{c} J$ such that:

    a. there exists a morphism $J \xrightarrow{d} D$ such that $f = d \circ c$;

    b. for any object J' with a morphism $J' \rightarrow d'\, D$ and any epic $C \xrightarrow{c'} J'$ such that $f = d' \circ c'$, there is a unique morphism $J' \xrightarrow{u} J$ such that $c = u \circ c'$.

    The object J is denoted by $\mathrm{Coim}(f)$ and the morphism c by $\mathrm{coim}(f)$.

    Show that in the category $\mathcal{A}b$ for a morphism $C \xrightarrow{f} D$ the pair consisting of the group $C/\mathrm{Ker}(f)$ canonical epimorphism is the coimage.

Homework: Problems 9, 10 and 11.