# 12 Morphisms of projective algebraic sets. Rational maps of projective algebraic sets.

## 12.1 Graded algebras.

**Definition 12.1.** *Let $k$ be any field. A* ***graded k-algebra*** *is a $k$-algebra $R$ such that*

$$R = R_0 \oplus R_1 \oplus \cdots \oplus R_i \oplus \cdots$$

*where $R_0 = k$, $R_i$ are vector spaces over $k$ and if $a \in R_i$ and $b \in R_j$, then $ab \in R_{i+j}$. An element $a \in R_i$ is called the* ***homogeneous element*** *of degree $i$.*

**Example 12.2.** Let $k$ be any field. Then

$$k[x_1, \ldots, x_n] = k[x_1, \ldots, x_n]_0 \oplus k[x_1, \ldots, x_n] \oplus \cdots \oplus k[x_1, \ldots, x_n]_i \oplus \cdots$$

where $k[x_1, \ldots, x_n]_i$ is the set of forms of degree $i$. Thus $k[x_1, \ldots, x_n]$ is a graded algebra.

**Remark 12.3.** Let $R = R_0 \oplus R_1 \oplus \cdots \oplus R_i \oplus \cdots$ be a graded $k$-algebra, let $\mathfrak{a} \lhd R$ be an ideal. Then the following two conditions are equivalent:

1. if $a = a_0 + a_1 + \cdots + a_n \in \mathfrak{a}$ with $a_i \in R_i$, then $a_i \in \mathfrak{a}$, for $i \in \{0, \ldots, n\}$;

2. $\mathfrak{a}$ is generated by homogenous elements.

**Proof.** 1. $\Rightarrow$ 2. is obvious.

2. $\Rightarrow$ 1. Let $a = a_0 + a_1 + \cdots + a_n \in \mathfrak{a}$ with $a_i \in R_i$, $i \in \{0, \ldots, n\}$. Since $\mathfrak{a}$ is generated by homogeneous coordinates, $a = b_0 + \cdots + b_m$ for some $b_i \in R_i$, $i \in \{0, \ldots, m\}$. But since $R$ is a direct sum of $R_i$'s, it follows that the presentation of $a$ as a sum of summands from $R_i$'s is unique, hence $n = m$ and $a_i = b_i$, $i \in \{0, \ldots, n\}$. □

**Definition 12.4.** *Let $R = R_0 \oplus R_1 \oplus \cdots \oplus R_i \oplus \cdots$ be a graded $k$-algebra. An ideal $\mathfrak{a} \lhd R$ is* ***homogeneous*** *if it satifies one of the two equivalent conditions of Remark 12.3.*

**Remark 12.5.** Let $R = R_0 \oplus R_1 \oplus \cdots \oplus R_i \oplus \cdots$ be a graded $k$-algebra, let $\mathfrak{a} \lhd R$ be a homogeneous ideal. Then

$$\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_i \oplus \cdots$$

where $\mathfrak{a}_i = R_i \cap \mathfrak{a}$. Moreover

$$R/\mathfrak{a} \cong R_0/\mathfrak{a}_0 \oplus R_1/\mathfrak{a}_1 \oplus \cdots \oplus R_i/\mathfrak{a}_i \oplus \cdots$$

is a graded $k$-algebra.

## 12.2 Regular functions.

Let $\mathbb{P}^n(k)$ be a projective $n$-space over $k$, let $P \in \mathbb{P}^n(k)$, let $f \in k[x_1, \ldots, x_{n+1}]$. If $P = [a_1 : \ldots : a_{n+1}]$ then $f(a_1, \ldots, a_{n+1}) \neq f(\lambda a_1, \ldots, \lambda a_{n+1})$ for $\lambda \neq 1$, so that the value of $P$ depends on the choice of the homogeneous coordinates of $P$. For this reason we restrain ourselves from defining the "coordinate ring" of a projective algebraic set, whose elements would correspond to polynomial functions having the same values at $V$.

Instead we focus of regular functions. Recall that a regular function on an affine variety $V$ was a rational function that was defined on some open subset of $V$. In particular, rational functions defined over all of $V$ corresponded with elements of the coordinate ring of $V$. We thus begin with defining rational functions on projective varietes.

**Remark 12.6.** Let $\mathbb{P}^n(k)$ be a projective $n$-space over $k$, let $V \subseteq \mathbb{P}^n(k)$ be an irreducible projective algebraic set. Let $f = F + \mathcal{I}(V), g = G + \mathcal{I}(V) \in k[x_1, ..., x_{n+1}]/\mathcal{I}(V)$ with $F, G \in k[x_1, ..., x_{n+1}]$. If $P \in V$, $P = [x_1 \colon ... \colon x_{n+1}]$, then $\frac{f}{g} \colon V \to k$ defined by

$$\frac{f}{g}(P) = \frac{F(x_1, ..., x_{n+1})}{G(x_1, ..., x_{n+1})}$$

is a function if $f$ is **homogenous** of degree 0, that is $F$ and $G$ are both forms of the same degree.

**Proof.** Indeed, say $F$ and $G$ are both forms of degree $d$. Then

$$\frac{F(\lambda x_1, ..., \lambda x_{n+1})}{G(\lambda x_1, ..., \lambda x_{n+1})} = \frac{\lambda^d F(x_1, ..., x_{n+1})}{\lambda^d G(x_1, ..., x_{n+1})} = \frac{F(x_1, ..., x_{n+1})}{G(x_1, ..., x_{n+1})}. \qquad \square$$

**Remark 12.7.** Let $\mathbb{P}^n(k)$ be a projective $n$-space over $k$, let $V \subseteq \mathbb{P}^n(k)$ be an irreducible projective algebraic set. Then

$$k(V) = \left\{ \frac{f}{g} \mid f = F + \mathcal{I}(V), g = G + \mathcal{I}(V) \in k[x_1, ..., x_{n+1}]/\mathcal{I}(V), F, G \text{ are forms of the same degree} \right\}$$

is a subfield of the field of fractions of the ring $k[x_1, ..., x_{n+1}]/\mathcal{I}(V)$.

**Definition 12.8.** *Let $\mathbb{P}^n(k)$ be a projective $n$-space over $k$, let $V \subseteq \mathbb{P}^n(k)$ be an irreducible projective algebraic set. The field $k(V)$ defined in Remark 12.7 is called the **rational function field** of V.*

*Let $V \subseteq k^n$ be a projective algebraic set, let $V = V_1 \cup ... \cup V_m$ be the decomposition of V into irreducible projective algebraic sets. The **k-algebra of rational functions** of V is defined to be*

$$k(V) = k(V_1) \oplus ... \oplus k(V_m)$$

*and its elements are called **rational functions** on V*

**Definition 12.9.** *Let $\mathbb{P}^n(k)$ be a projective $n$-space over $k$, let $V \subseteq \mathbb{P}^n(k)$ be an irreducible projective algebraic set. A rational function $\varphi \in k(V)$ is **defined** at a point $[a_1 \colon ... \colon a_{n+1}] \in V$ if $\varphi = \frac{f}{g}$, for some $f = F + \mathcal{I}(V)$, $g = G + \mathcal{I}(V) \in k[x_1, ..., x_{n+1}]/\mathcal{I}(V)$, $F, G \in k[x_1, ..., x_n]$, with $G(a_1, ..., a_n) \neq 0$. In this case we say that $\frac{F(a_1, ..., a_{n+1})}{G(a_1, ..., a_{n+1})} \in k$ is the **value** of $\varphi$ at $P = [a_1 \colon ... \colon a_{n+1}]$, and denote it by $\varphi(P)$.*

**Remark 12.10.** Let $\mathbb{P}^n(k)$ be a projective $n$-space over $k$, let $V \subseteq \mathbb{P}^n(k)$ be an irreducible projective algebraic set, let $\varphi \in k(V)$ be defined at $P \in V$. The value of $\varphi$ at $P$ is uniquely defined.

The proof is almost identical with the proof of Remark 10.4 and we leave it as an excercise.

**Remark 12.11.** Let $\mathbb{P}^n(k)$ be a projective $n$-space over $k$, let $V \subseteq \mathbb{P}^n(k)$ be an irreducible projective algebraic set. Every rational function $\frac{f}{g} \in k(V)$, $f = F + \mathcal{I}(V), g = G + \mathcal{I}(V) \in k[V]$, $F$, $G \in k[x_1, ..., x_{n+1}]$ determines a function defined on some nonempty open subset $U \subseteq V$ with values in $k$ that we shall also call a rational function.

The proof is almost identical with the proof of Remark 10.6 and we leave it as an excercise.

**Remark 12.12.** Let $\mathbb{P}^n(k)$ be a projective $n$-space over an algebraically closed field $k$, let $V \subseteq \mathbb{P}^n(k)$ be an irreducible projective algebraic set. If the rational function $\varphi \in k(V)$ is defined at every point of $V$, then $\varphi \in k[x_1, ..., x_{n+1}]/\mathcal{I}(V)$.

The proof is almost identical with the proof of Theorem 10.8 and we leave it as an excercise.

**Definition 12.13.** *Let $\mathbb{P}^n(k)$ be a projective $n$-space over $k$, let $V \subseteq \mathbb{P}^n(k)$ be an irreducible projective algebraic set. If a rational function $\varphi \in k(V)$ is defined at every point of an open subset $U \subseteq V$, then the restriction $\varphi \restriction_U$ will be called a **regular function on U**. The set of all regular functions on $U$ will be denoted by $\mathcal{O}_V(U)$. Functions regular on $V$ will be called **regular** and the set of all regular functions on $V$ denoted by $k[V]$.*

**Proposition 12.14.** *Let $\mathbb{P}^n(k)$ be a projective $n$-space over an algebraically closed field $k$. Then $k[\mathbb{P}^n(k)] \cong k$, that is the only functions defined at all points of $\mathbb{P}^n(k)$ are constants.*

**Proof.** The ring $k[x_1, ..., x_{n+1}]$ is a unique factorization domain, hence every $\varphi \in k(\mathbb{P}^n(k))$ can be represented as $\frac{f}{g}$ with$f = F + \mathcal{I}(\mathbb{P}^n(k))$, $g = G + \mathcal{I}(\mathbb{P}^n(k)) \in k[x_1, ..., x_{n+1}] + \mathcal{I}(\mathbb{P}^n(k))$ and $\gcd(F, G) = 1$. Thus any other representation of $\varphi$ is of the form $\frac{hf}{hg}$, $h = H + \mathcal{I}(\mathbb{P}^n(k)) \in k[x_1, ..., x_{n+1}] + \mathcal{I}(\mathbb{P}^n(k))$ and hence it is not defined at every point $P \in \mathbb{P}^n(k)$ such that $G(P) = 0$. Since every form of positive degree $d$ is equal to zero at some $(a_1, ..., a_{n+1}) \neq (0, ..., 0)$ (if $G \neq x_i^d$, take $x_i = 1$ and use the Nullstellensatz), the set of points where $\varphi$ is not defined is nonempty as long, as $f$ and $g$ contain as summands forms of positive degree. This would lead to a contradiction, and the only forms that do not contains as summands forms of positive degree are constants. ☐

**Proposition 12.15.** *Let $\mathbb{P}^n(k)$ be a projective $n$-space over an algebraically closed field $k$, let $V \subseteq \mathbb{P}^n(k)$ be an irreducible projective algebraic set. Then $k[V] \cong k$, that is the only functions defined at all points of $V$ are constants.*

**Proof.** Let $\varphi \in k[V]$. Write $\varphi = \frac{f}{g}$ with $f = F + \mathcal{I}(V), g = G + \mathcal{I}(V) \in k[x_1, ..., x_{n+1}] + \mathcal{I}(V)$, where $F$ and $G$ are forms of the same degree. Let $\tilde{V}$ be an affine algebraic set such that $k[\tilde{V}] \cong k[x_1, ..., x_{n+1}]/\mathcal{I}(V)$. Then $(0, ..., 0) \in \tilde{V}$ and $\varphi$ determines a rational function on $\tilde{V}$ defined everywhere but at $(0, ..., 0)$. Thus, for $i \in \{1, ..., n+1\}$, there exists an integer $n_i$ such that $x_i^{n_i}\varphi \in k[x_1, ..., x_{n+1}]/\mathcal{I}(V)_{n_i}$. We shall show that there exists $m$ such that

$$\varphi k[x_1, ..., x_{n+1}]/\mathcal{I}(V)_s \subseteq k[x_1, ..., x_{n+1}]/\mathcal{I}(V)_s, \text{ for every } s \geqslant m.$$

Indeed, let $m = n_1 + \cdots + n_{n+1}$, $s \geqslant m$ and $h \in k[x_1, ..., x_{n+1}]/\mathcal{I}(V)_s$. It suffices to show that $h\varphi \in k[x_1, ..., x_{n+1}]/\mathcal{I}(V)_s$. We might as well assume that $h = x_1^{m_1} \cdot ... \cdot x_{n+1}^{m_{n+1}}$, since every element of $k[x_1, ..., x_{n+1}]/\mathcal{I}(V)$ is a linear combination of such elements. Since, for some $i \in \{1, ..., n+1\}$, we have that $m_i \geqslant n_i$, and hence $h\varphi \in k[x_1, ..., x_{n+1}]/\mathcal{I}(V)$. But $\varphi$ is a quotient of two homogeneous elements of the same degree, so $h$ and $h\varphi$ are of the same degree, and hence $h\varphi \in k[x_1, ..., x_{n+1}]/\mathcal{I}(V)_s$.

By induction, $\varphi^t k[x_1, ..., x_{n+1}]/\mathcal{I}(V)_s \subseteq k[x_1, ..., x_{n+1}]/\mathcal{I}(V)_s$ for every $t$ and $s \geqslant m$. Denote by $k[x_1, ..., x_{n+1}]/\mathcal{I}(V)_{>m}$ the union of all $k[x_1, ..., x_{n+1}]/\mathcal{I}(V)_s$ in $k[x_1, ..., x_{n+1}]/\mathcal{I}(V)$ with $s > m$. Then $k[x_1, ..., x_{n+1}]/\mathcal{I}(V)_{>m}$ is a finitely generated submodule of $k[x_1, ..., x_{n+1}]/\mathcal{I}(V)$ and $\varphi k[x_1, ..., x_{n+1}]/\mathcal{I}(V)_{>m} \subseteq k[x_1, ..., x_{n+1}]/\mathcal{I}(V)_{>m}$. Thus the element of the field of fractions of $k[x_1, ..., x_{n+1}]/\mathcal{I}(V)$ is integral over $k[x_1, ..., x_{n+1}]/\mathcal{I}(V)$. Let $\xi(x) = x^r + a_{r-1}x^{r-1} \cdots + a_0 \in k[x_1, ..., x_{n+1}]/\mathcal{I}(V)[x]$ be the irreducible monic polynomial whose zero is $\varphi$. As $\varphi$ is a quotient of homogeneous elements of the same degree, the polynomial obtained by replacing in $\xi$ every $a_j$ by its summand of degree 0 is a nonzero polynomial from the ring $k[x]$, whose root is $\varphi$. But $k$ is algebraically closed, and hence $\varphi \in k$. ☐

## 12.3 Morphisms of projective algebraic sets.

**Definition 12.16.** *Let $\mathbb{P}^n(k)$ be a projective $n$-space over a field $k$, let $V \subseteq \mathbb{P}^n(k)$ be a quasiprojective algebraic variety (that is, an open subset of some projective algebraic set $V' \subseteq \mathbb{P}^n(k)$). A **morphism** $f: V \to k^m$ is a map such that there exist $f_1, \ldots, f_m \in k[V]$ such that $f(a) = (f_1(a), \ldots, f_m(a))$, for all $a \in V$.*

*Let $\mathbb{P}^m(k)$ be a projective $m$-space over a field $k$, let $W \subseteq \mathbb{P}^m(k)$ be a quasiprojective algebraic variety. Recall that $\mathbb{P}^m(k) = \bigcup_{i=1}^{m+1} U_i$, where*

$$U_i = \{[y_1: \ldots: y_{n+1}] \in \mathbb{P}^n(k) | \, y_i \neq 0\}$$

*is in bijective correspondence with $k^m$ via the map $\varphi_i: k^m \to U_i$ given by*

$$\varphi_i(x_1, \ldots, x_m) = [x_1: \ldots: x_{i-1}: 1: x_{i+1}: \ldots: x_m].$$

*A **morphism** $f: V \to W$ is a map such that for every $P \in V$, if $f(P) \in U_i$, for some $i \in \{1, \ldots, m+1\}$, then there is a neighbourhood $U_P \subseteq V$ of $P$ such that $f(U_P) \subseteq U_i$ and $f \restriction_{U_P}: U_P \to U_i$ is a morphism.*

**Remark 12.17.** Morphisms of quasiprojective varietes are well-defined, that is the definition does not depend on the choice of $U_i$.

**Proof.** Let $\mathbb{P}^n(k)$, $\mathbb{P}^m(k)$ be projective spaces over a field $k$, let $V \subseteq \mathbb{P}^n(k)$ and $W \subseteq \mathbb{P}^m(k)$ be quasiprojective varietesm let $f: V \to W$ be a fixed function. Say that for a fixed $P \in V$ we have $f(P) \in U_i$, for some $i \in \{1, \ldots, m+1\}$. Then $f(P) = [y_1: \ldots: y_{i-1}: 1: y_{i+1}: \ldots: y_m]$. If $f(P) \in U_j$, for some $i \neq j$, then $y_j \neq 0$, and $[y_1: \ldots: y_{i-1}: 1: y_{i+1}: \ldots: y_m] = \left[\frac{y_0}{y_1}: \ldots: \frac{y_{i-1}}{y_j}: \frac{1}{y_j}: \frac{y_{i+1}}{y_j}: \ldots: \frac{y_{j-1}}{y_j}: 1: \frac{y_{j+1}}{y_j} \ldots: \frac{y_m}{y_j}\right]$. Thus if the morphism $f: U_P \to U_i$ is given by regular functions $(f_1, \ldots, f_{i-1}, 1, f_{i+1}, \ldots, f_m)$, then the morphism given by the functions $\left(\frac{f_1}{f_j}, \ldots, \frac{1}{f_j}, \ldots, \frac{f_m}{f_j}\right)$ maps to $U_j$ from the subset $U_P' \subseteq U_P$ of those points, where $f_j(P) \neq 0$, and the set $U_P'$ is clearly open. □