

## 11 Projective space. Projective algebraic sets.

### 11.1 Projective space.

The concept of a projective space originated from the visual effect of perspective, where parallel lines seem to meet at infinity. A projective space may thus be viewed as the extension of an Euclidean space in such a way that there is one point at infinity for each direction of parallel lines.

Consider the affine plane  $k^2$ . Each point  $(x, y) \in k^2$  can be identified with the point  $(x, y, 1) \in k^3$ . Every point  $(x, y, 1) \in k^3$  determines a line in  $k^3$  that passes through  $(0, 0, 0)$  and  $(x, y, 1)$ . Every line through  $(0, 0, 0)$  except those lying on the plane  $z = 0$  corresponds to exactly one such point. The lines through  $(0, 0, 0)$  in the plane  $z = 0$  can be thought of as corresponding to the “points at infinity”.

**Definition 11.1.** Let  $k$  be a field. **Projective  $n$ -space over  $k$** , written  $\mathbb{P}^n(k)$ , is defined to be the set of all lines through  $(0, \dots, 0) \in k^{n+1}$ . Any point  $(x_1, \dots, x_{n+1}) \neq (0, \dots, 0)$  determines a unique such line, namely  $\{(\lambda x_1, \dots, \lambda x_{n+1}) \mid \lambda \in k\}$ . Elements of  $\mathbb{P}^n(k)$  are called **points**. If a point  $P \in \mathbb{P}^n(k)$  is determined as above by some  $(x_1, \dots, x_{n+1}) \in k^{n+1} \setminus \{(0, \dots, 0)\}$ , we say that  $(x_1, \dots, x_{n+1})$  are the **homogeneous coordinates** of  $P$  and write  $P = [x_1 : \dots : x_{n+1}]$ .

**Remark 11.2.** Let  $\mathbb{P}^n(k)$  be a projective  $n$ -space over  $k$ . Two points  $(x_1, \dots, x_{n+1}), (y_1, \dots, y_{n+1}) \in k^{n+1}$  determine the same line if and only if there is a nonzero  $\lambda \in k$  such that  $x_1 = \lambda y_1, \dots, x_{n+1} = \lambda y_{n+1}$ . Let us say that  $(x_1, \dots, x_{n+1}), (y_1, \dots, y_{n+1}) \in k^{n+1}$  are equivalent if this is the case. Then  $\mathbb{P}^n(k)$  may be identified with the set of equivalence classes of points  $k^{n+1} \setminus \{(0, \dots, 0)\}$ .

**Remark 11.3.** Let  $\mathbb{P}^n(k)$  be a projective  $n$ -space over  $k$  and let  $P \in \mathbb{P}^n(k)$ ,  $P = [x_1 : \dots : x_{n+1}]$ . Note that the  $i$ -th coordinate  $x_i$  is not well-defined, but that it is a well-defined notion to say whether the  $i$ -th coordinate is zero or nonzero. If  $x_i \neq 0$  the ratios  $x_j/x_i$  are well-defined, since they are unchanged under the abovedescribed equivalence.

**Definition 11.4.** Let  $\mathbb{P}^n(k)$  be a projective  $n$ -space over  $k$ . Let

$$U_i = \{[y_1 : \dots : y_{n+1}] \in \mathbb{P}^n(k) \mid y_i \neq 0\}$$

Each  $P \in U_i$  can be written uniquely in the form

$$P = [x_1 : \dots : x_{i-1} : 1 : x_{i+1} : \dots : x_{n+1}].$$

The coordinates  $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1})$  are called the **nonhomogeneous coordinates** of  $P$  with respect to  $U_i$ .

**Remark 11.5.** Let  $\mathbb{P}^n(k)$  be a projective  $n$ -space over  $k$ . Define  $\varphi_i: k^n \rightarrow U_i$  by

$$\varphi_i(x_1, \dots, x_n) = [x_1 : \dots : x_{i-1} : 1 : x_{i+1} : \dots : x_n].$$

Then  $\varphi_i$  defines a bijective correspondence between  $k^n$  and  $U_i$ . Note that  $\mathbb{P}^n(k) = \bigcup_{i=1}^{n+1} U_i$ , so that  $\mathbb{P}^n(k)$  is covered by  $n+1$  bijective copies of  $k^n$ .

**Definition 11.6.** Let  $\mathbb{P}^n(k)$  be a projective  $n$ -space over  $k$ . The set

$$H_\infty = \mathbb{P}^n(k) \setminus U_{n+1} = \{[x_1 : \dots : x_{n+1}] \mid x_{n+1} = 0\}$$

is called the **hyperplane at infinity**.

**Remark 11.7.** Let  $\mathbb{P}^n(k)$  be a projective  $n$ -space over  $k$ . The map  $H_\infty \rightarrow \mathbb{P}^{n-1}(k)$  given by

$$[x_1: \dots: x_n: 0] \mapsto [x_1: \dots: x_n]$$

is bijective. Thus  $H_\infty$  may be identified with  $\mathbb{P}^{n-1}(k)$  and  $\mathbb{P}^n(k) = U_{n+1} \cup H_\infty$  is the union of an affine  $n$ -space and a set that gives all directions in affine  $n$ -space.

**Example 11.8.**

1.  $\mathbb{P}^0(k)$  is a point.
2.  $\mathbb{P}^1(k) = \{[x: 1] \mid x \in k\} \cup \{[1: 0]\}$  is the affine line plus one point at infinity. We call it **projective line** over  $k$ .
3.  $\mathbb{P}^2(k) = \{[x: y: 1] \mid x, y \in k\} \cup \{[x: y: 0] \mid [x: y] \in \mathbb{P}^1(k)\}$ . Here  $H_\infty$  is called the **line at infinity**.  $\mathbb{P}^2(k)$  is called the **projective plane** over  $k$ .
4. Consider a line  $\ell: y = ax + b$  in  $k^2$ . If we identify  $k^2$  with  $U_3 \subseteq \mathbb{P}^2(k)$ , the points on the line  $\ell$  correspond to the points  $\{[x: y: z] \mid y = ax + bz \text{ and } z \neq 0\} \in \mathbb{P}^2(k)$ . Then

$$\{[x: y: z] \mid y = ax + bz\} \cap H_\infty = \{[1: a: 0]\},$$

so that all lines with the same slope  $a$ , when extended that way, pass through the same point at infinity.

5. Consider the curve  $C: y^2 = x^2 + 1$  in  $k^2$ . The corresponding set in  $\mathbb{P}^2(k)$  is given by the equation  $y^2 = x^2 + z^2$ ,  $z \neq 0$ . Thus

$$\{[x: y: z] \mid y^2 = x^2 + z^2\} \cap H_\infty = \{[1: 1: 0], [1: -1: 0]\}.$$

These are the points where the lines  $y = x$  and  $y = -x$  intersect the curve.

## 11.2 Projective algebraic sets.

**Definition 11.9.** Let  $\mathbb{P}^n(k)$  be a projective  $n$ -space over  $k$ . A point  $P \in \mathbb{P}^n(k)$  is said to be a **zero** of a polynomial  $f \in k[x_1, \dots, x_{n+1}]$  if  $f(x_1, \dots, x_{n+1}) = 0$  for every choice of homogenous coordinates for  $P$ ; we then write  $f(P) = 0$ .

**Definition 11.10.** A polynomial  $f \in k[x_1, \dots, x_{n+1}]$  is called a **form** of degree  $d$  if it is a sum of monomials of degree  $d$ :

$$f = \sum_{(i_1, \dots, i_{n+1}) \in S \subseteq \mathbb{N}^{n+1}} a_{i_1 \dots i_{n+1}} x_1^{i_1} \dots x_{n+1}^{i_{n+1}}, \quad i_1 + \dots + i_{n+1} = d.$$

**Remark 11.11.** A polynomial  $f \in k[x_1, \dots, x_{n+1}]$  is a form of degree  $d$  if and only if

$$f(ab_1, \dots, ab_{n+1}) = a^d f(b_1, \dots, b_{n+1}),$$

for all  $a, b_1, \dots, b_{n+1} \in k$ .

**Proof.** If  $f = \sum_{(i_1, \dots, i_{n+1}) \in S \subseteq \mathbb{N}^{n+1}} a_{i_1 \dots i_{n+1}} x_1^{i_1} \dots x_{n+1}^{i_{n+1}}$  where  $i_1 + \dots + i_{n+1} = d$  for all  $(i_1, \dots, i_n) \in S$ , then for any  $a, b_1, \dots, b_{n+1} \in k$ :

$$\begin{aligned} f(ab_1, \dots, ab_{n+1}) &= \sum_{(i_1, \dots, i_{n+1}) \in S \subseteq \mathbb{N}^{n+1}} a_{i_1 \dots i_{n+1}} (ab_1)^{i_1} \dots (ab_{n+1})^{i_{n+1}} \\ &= \sum_{(i_1, \dots, i_{n+1}) \in S \subseteq \mathbb{N}^{n+1}} a_{i_1 \dots i_{n+1}} a^{i_1 + \dots + i_{n+1}} b_1^{i_1} \dots b_{n+1}^{i_{n+1}} \\ &= a^d \sum_{(i_1, \dots, i_{n+1}) \in S \subseteq \mathbb{N}^{n+1}} a_{i_1 \dots i_{n+1}} b_1^{i_1} \dots b_{n+1}^{i_{n+1}} \\ &= a^d f(b_1, \dots, b_{n+1}). \end{aligned}$$

Conversely, assume that  $f(ab_1, \dots, ab_{n+1}) = a^d f(b_1, \dots, b_{n+1})$ , for all  $a, b_1, \dots, b_{n+1} \in k$ . Let  $f = \sum_{(i_1, \dots, i_{n+1}) \in S \subseteq \mathbb{N}^{n+1}} a_{i_1 \dots i_{n+1}} x_1^{i_1} \dots x_{n+1}^{i_{n+1}}$ . Then

$$f(ab_1, \dots, ab_{n+1}) = \sum_{(i_1, \dots, i_{n+1}) \in S \subseteq \mathbb{N}^{n+1}} a_{i_1 \dots i_{n+1}} a^{i_1 + \dots + i_{n+1}} b_1^{i_1} \dots b_{n+1}^{i_{n+1}}.$$

On the other hand

$$a^d f(b_1, \dots, b_{n+1}) = \sum_{(i_1, \dots, i_{n+1}) \in S \subseteq \mathbb{N}^{n+1}} a_{i_1 \dots i_{n+1}} a^d b_1^{i_1} \dots b_{n+1}^{i_{n+1}},$$

which yields  $a^{i_1 + \dots + i_{n+1}} = a^d$ , and, consequently,  $i_1 + \dots + i_{n+1} = d$  for all  $(i_1, \dots, i_n) \in S$ .  $\square$

**Remark 11.12.** Let  $\mathbb{P}^n(k)$  be a projective  $n$ -space over  $k$ , let  $P \in \mathbb{P}^n(k)$ . If  $f \in k[x_1, \dots, x_{n+1}]$  is a form of degree  $d$  and  $f$  vanishes at one representative of  $P$ , then it vanishes at every representative of  $P$ .

**Definition 11.13.** Let  $\mathbb{P}^n(k)$  be a projective  $n$ -space over  $k$ . A **projective algebraic set**  $V$  is a subset of the projective  $n$ -space  $\mathbb{P}^n(k)$  consisting of all common zeros of some set of forms  $S \subseteq k[x_1, \dots, x_{n+1}]$ :

$$V = \{[a_1 : \dots : a_{n+1}] \in \mathbb{P}^n(k) \mid f(a_1, \dots, a_{n+1}) = 0 \text{ for all } f \in S\}.$$

We shall call the set  $V$  to be defined by the set of forms  $S$  and denote by  $V = \mathcal{Z}(S)$ .

**Definition 11.14.** An ideal  $\mathfrak{a} \triangleleft k[x_1, \dots, x_{n+1}]$  is called **homogeneous** if for every  $f \in \mathfrak{a}$ , if

$$f = \sum_{d=0}^m f^{(d)},$$

where  $f^{(d)}$  is a form of degree  $d$ , then also  $f^{(0)}, \dots, f^{(m)} \in \mathfrak{a}$ .

**Proposition 11.15.** An ideal  $\mathfrak{a} \triangleleft k[x_1, \dots, x_{n+1}]$  is homogeneous if and only if it is generated by a finite set of forms.

**Proof.** Assume that  $\mathfrak{a} \triangleleft k[x_1, \dots, x_{n+1}]$  is homogeneous. Since  $k[x_1, \dots, x_{n+1}]$  is Noetherian,  $\mathfrak{a} = \langle f_1, \dots, f_k \rangle$ , for some  $f_1, \dots, f_k \in k[x_1, \dots, x_{n+1}]$ . Write  $f_i = \sum_{d=0}^{m_i} f_i^{(d)}$ , where  $f_i^{(d)}$  is a form of degree  $d$ ,  $i \in \{1, \dots, k\}$ . Then  $f_i^{(d)} \in \mathfrak{a}$ , as  $\mathfrak{a}$  is homogeneous, and hence  $\langle f_i^{(d)} \mid i \in \{1, \dots, k\}, d \in \{0, \dots, \max\{m_1, \dots, m_k\}\} \rangle \subseteq \mathfrak{a}$ . But as every element of  $\mathfrak{a}$  is a combination of  $f_1, \dots, f_k$  which, in turn, are combinations of  $f_i^{(d)} \mid i \in \{1, \dots, k\}, d \in \{0, \dots, \max\{m_1, \dots, m_k\}\}$ , the other inclusion also holds.

Conversely, assume that  $\mathfrak{a} = \langle f_1^{(d_1)}, \dots, f_k^{(d_k)} \rangle$ , where  $f_i^{(d_i)} \in k[x_1, \dots, x_{n+1}]$  is a form of degree  $d_i$ . Let  $f \in \mathfrak{a}$ . Write  $f = \sum_{i=m}^r f_i$ , where  $\deg f_i = i$ . It suffices to show that  $f_m \in \mathfrak{a}$ , for then  $f - f_m \in \mathfrak{a}$  and an inductive argument finishes the proof. Write  $f = \sum_{i=1}^k a_i \cdot f_i^{(d_i)}$ , for some  $a_1, \dots, a_k \in k[x_1, \dots, x_{n+1}]$ . Comparing terms with the same degree we conclude that  $f_m = \sum_{i \in \{i | d_i = m\}} a_i \cdot f_i^{(d_i)}$ , so  $f_m \in \mathfrak{a}$ .  $\square$

**Remark 11.16.** Let  $\mathcal{S} \subseteq k[x_1, \dots, x_{n+1}]$  be a set of forms and let  $\mathfrak{a}$  be the homogenous ideal of  $k[x_1, \dots, x_{n+1}]$  generated by  $\mathcal{S}$ . Then

$$\mathcal{Z}(\mathcal{S}) = \mathcal{Z}(\mathfrak{a}).$$

**Remark 11.17.** Let  $\mathcal{S} \subseteq k[x_1, \dots, x_{n+1}]$  be a set of forms. Then there exists a finite set of forms  $\{f_1, \dots, f_r\} \subseteq k[x_1, \dots, x_{n+1}]$  such that

$$\mathcal{Z}(\mathcal{S}) = \mathcal{Z}(f_1, \dots, f_r).$$

**Remark 11.18.** Let  $\mathbb{P}^n(k)$  be a projective  $n$ -space over  $k$ , let  $V \subseteq \mathbb{P}^n(k)$  be a projective algebraic set. The set  $\mathcal{I}(V)$  of all polynomials whose common zeros coincide with  $V$ :

$$\mathcal{I}(V) = \{f \in k[x_1, \dots, x_{n+1}] \mid f(a_1, \dots, a_{n+1}) = 0 \text{ for all } [a_1 : \dots : a_{n+1}] \in V\}$$

is a homogenous ideal of  $k[x_1, \dots, x_n]$ .

**Definition 11.19.** Let  $\mathbb{P}^n(k)$  be a projective  $n$ -space over  $k$ , let  $V \subseteq \mathbb{P}^n(k)$  be a projective algebraic set. The ideal  $\mathcal{I}(V)$  consisting of polynomials whose common zeros constitute  $V$  shall be called the **ideal of the projective algebraic set**  $V$ .

**Remark 11.20.** Let  $\mathbb{P}^n(k)$  be a projective  $n$ -space over  $k$ , let  $V, V_1, V_2 \subseteq \mathbb{P}^n(k)$  be projective algebraic sets, let  $\mathfrak{a}, \mathfrak{a}_1, \mathfrak{a}_2$  be homogenous ideals of  $k[x_1, \dots, x_{n+1}]$ . Then:

1.  $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \Rightarrow \mathcal{Z}(\mathfrak{a}_1) \supseteq \mathcal{Z}(\mathfrak{a}_2)$ ,
2.  $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) \supseteq \mathfrak{a}$ ,
3.  $\mathcal{Z}(\mathcal{I}(V)) = V$ ,
4.  $V_1 \subseteq V_2 \Leftrightarrow \mathcal{I}(V_1) \supseteq \mathcal{I}(V_2)$ ,
5.  $V_1 = V_2 \Leftrightarrow \mathcal{I}(V_1) = \mathcal{I}(V_2)$ .

### 11.3 Irreducible projective algebraic sets.

**Definition 11.21.** Let  $\mathbb{P}^n(k)$  be a projective  $n$ -space over  $k$ . A nonempty projective algebraic set  $V \subseteq \mathbb{P}^n(k)$  will be called **irreducible**, if for projective algebraic sets  $A, B \subseteq \mathbb{P}^n(k)$ :

$$V = A \cup B \Rightarrow V = A \vee B = B.$$

**Theorem 11.22.** Let  $\mathbb{P}^n(k)$  be a projective  $n$ -space over  $k$ . A nonempty projective algebraic set  $V \subseteq \mathbb{P}^n(k)$  is irreducible if and only if  $\mathcal{I}(V)$  is prime.

**Theorem 11.23.** Let  $\mathbb{P}^n(k)$  be a projective  $n$ -space over  $k$ . Every projective algebraic set  $V$  is a finite sum of irreducible projective algebraic sets:

$$V = V_1 \cup \dots \cup V_r, \quad r \geq 1.$$

If in the above decomposition the sets  $V_i$  are incomparable (that is  $V_i \not\subseteq V_j$  for  $i \neq j$ ), then they are uniquely defined.

## 11.4 Projective Nullstellensatz.

**Definition 11.24.** Let  $\mathbb{P}^n(k)$  be a projective  $n$ -space over  $k$ , let  $V \subseteq \mathbb{P}^n(k)$  be a projective algebraic set. The set

$$\mathcal{C}(V) = \{(x_1, \dots, x_{n+1}) \in k^n \mid [x_1 : \dots : x_n] \in V\} \cup \{(0, \dots, 0)\}$$

will be called the **cone** over  $V$ .

**Notation 11.25.** To avoid confusion when necessary, we shall write  $\mathcal{Z}_a(I)$  and  $\mathcal{I}_a(V)$  for affine operations and  $\mathcal{Z}_p(I)$  and  $\mathcal{I}_p(V)$  for projective operations.

**Remark 11.26.** Let  $\mathbb{P}^n(k)$  be a projective  $n$ -space over  $k$ , let  $V \subseteq \mathbb{P}^n(k)$  be a nonempty projective algebraic set. Then

$$\mathcal{I}_a(\mathcal{C}(V)) = \mathcal{I}_p(V).$$

Moreover, let  $\mathfrak{a} \triangleleft k[x_1, \dots, x_{n+1}]$  be a homogeneous ideal such that  $\mathcal{Z}_p(\mathfrak{a}) \neq \emptyset$ . Then

$$\mathcal{C}(\mathcal{Z}_p(\mathfrak{a})) = \mathcal{Z}_a(\mathfrak{a}).$$

**Corollary 11.27. (projective Nullstellensatz)** Let  $k$  be algebraically closed, let  $\mathfrak{a} \triangleleft k[x_1, \dots, x_{n+1}]$  be a homogeneous ideal. Then:

1.  $\mathcal{Z}_p(\mathfrak{a}) = \emptyset$  if and only if there is an integer  $N$  such that  $\mathfrak{a}$  contains all forms of degree  $\geq N$ ;
2. if  $\mathcal{Z}_p(\mathfrak{a}) \neq \emptyset$ , then  $\mathcal{I}_p(\mathcal{Z}_p(\mathfrak{a})) = \text{rad}(\mathfrak{a})$ .

**Proof.**

1. The following four conditions are equivalent:

- i.  $\mathcal{Z}_p(\mathfrak{a}) = \emptyset$ ,
- ii.  $\mathcal{Z}_p(\mathfrak{a}) \subseteq \{(0, \dots, 0)\}$ ,
- iii.  $\text{rad}(\mathfrak{a}) = \mathcal{I}_a(\mathcal{Z}_a(\mathfrak{a})) \supseteq \langle x_1, \dots, x_{n+1} \rangle$  (by the affine Nullstellensatz),
- iv.  $\langle x_1, \dots, x_{n+1} \rangle^N \subseteq \mathfrak{a}$ .

2.  $\mathcal{I}_p(\mathcal{Z}_p(\mathfrak{a})) = \mathcal{I}_a(\mathcal{C}(\mathcal{Z}_p(\mathfrak{a}))) = \mathcal{I}_a(\mathcal{Z}_a(\mathfrak{a})) = \text{rad}(\mathfrak{a})$ . □

## 11.5 Zariski topology. Quasiprojective varieties.

**Lemma 11.28.** A finite sum of projective algebraic sets is a projective algebraic set. To be more precise, let  $\mathfrak{a}_1, \dots, \mathfrak{a}_m$  be homogenous ideals of the ring  $k[x_1, \dots, x_{n+1}]$ . Then

$$\mathcal{Z}(\mathfrak{a}_1) \cup \dots \cup \mathcal{Z}(\mathfrak{a}_m) = \mathcal{Z}(\mathfrak{a}_1 \cdot \dots \cdot \mathfrak{a}_m),$$

where  $\mathfrak{a}_1 \cdot \dots \cdot \mathfrak{a}_m = \{\sum_{i=1}^k a_{i1}a_{i2}\dots a_{im} \mid k \in \mathbb{N}, a_{ij} \in \mathfrak{a}_j, j \in \{1, \dots, m\}, i \in \{1, \dots, k\}\}$ .

**Remark 11.29.** Let  $\mathfrak{a}_1, \dots, \mathfrak{a}_m$  be homogenous ideals of the ring  $k[x_1, \dots, x_{n+1}]$ . Then

$$\mathcal{Z}(\mathfrak{a}_1 \cdot \dots \cdot \mathfrak{a}_m) = \mathcal{Z}(\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_m).$$

**Lemma 11.30.** *Intersection of any number of projective algebraic sets is a projective algebraic set. To be more precise, let  $\{\mathfrak{a}_i | i \in I\}$  be a family of homogenous ideals of the ring  $k[x_1, \dots, x_{n+1}]$ . Then*

$$\bigcap_{i \in I} \mathcal{Z}(\mathfrak{a}_i) = \mathcal{Z}\left(\left(\bigcup_{i \in I} \mathfrak{a}_i\right)\right).$$

**Remark 11.31.** Let  $\mathfrak{a}_1, \dots, \mathfrak{a}_m$  be homogenous ideals of the ring  $k[x_1, \dots, x_{n+1}]$ . Then

$$\mathcal{Z}(\mathfrak{a}_1 + \dots + \mathfrak{a}_m) = \mathcal{Z}(\langle \mathfrak{a}_1 \cup \dots \cup \mathfrak{a}_m \rangle).$$

**Theorem 11.32.** *In  $\mathbb{P}^n(k)$  there is a topology whose closed sets are projective algebraic sets in  $\mathbb{P}^n(k)$ .*

**Definition 11.33.** *The topology of  $\mathbb{P}^n(k)$  defined by projective algebraic sets is called the **Zariski topology** in  $\mathbb{P}^n(k)$ .*

**Definition 11.34.** *Open subsets of closed projective algebraic sets are called **quasiprojective varieties**.*