10 Rational functions field of an affine algebraic variety. Rational maps of affine algebraic sets. Birational equivalence of affine algebraic sets.

Definition 10.1. *Let $V\subseteq k^n$ be an affine algebraic variety. The field of fractions of the coordinate ring $k[V]$ will be called the **field of rational functions** of V and denoted by $k(V)$, and its elements **rational functions** on V.*

Example 10.2. Consider the following easy examples:

- $V=\{(a_1,...,a_n)\}\in k^n$, $k(V)\cong k$;
- $V=k^k$, $k(V)\cong k(x_1,...,x_n)$.

Definition 10.3. *Let $V\subseteq k^n$ be an affine algebraic variety. A rational function $\varphi\in k(V)$ is **defined** at a point $(a_1,...,a_n)\in V$ if $\varphi=\frac{f}{g}$, for some $f=F+\mathcal{I}(V), g=G+\mathcal{I}(V)\in k[V]$, $F,G\in k[x_1,...,x_n]$, with $G(a_1,...,a_n)\neq 0$. In this case we say that $\frac{F(a_1,...,a_n)}{G(a_1,...,a_n)}\in k$ is the **value** of φ at $(a_1,...,a_n)$, and denote it by $\varphi(a_1,...,a_n)$.*

Remark 10.4. Let $V\subseteq k^n$ be an affine algebraic variety, let $\varphi\in k(V)$ be defined at $(a_1,...,a_n)\in V$. The value of φ at $(a_1,...,a_n)$ is uniquely defined.

Proof. Let $\varphi=\frac{f_1}{g_1}=\frac{f_2}{g_2}$, $f_1=F_1+\mathcal{I}(V), f_2=F_2+\mathcal{I}(V), g_1=G_1+\mathcal{I}(V), g_2=G_2+\mathcal{I}(V)\in k[V]$, F_1, $F_2, G_1, G_2\in k[x_1, ..., x_n]$, with $G_1(a_1, ..., a_n)\neq 0$ and $G_2(a_1, ..., a_n)\neq 0$ be two presentations of φ as a quotient of elements of the coordinate ring of V. Then $f_1g_2=f_2g_1$ in the ring $k[V]$, so that $F_1(a_1,...,a_n)G_2(a_1,...,a_n)=F_2(a_1,...,a_n)G_1(a_1,...,a_n)$ and thus

$$\frac{F_1(a_1,...,a_n)}{G_1(a_1,...,a_n)}=\frac{F_2(a_1,...,a_n)}{G_2(a_1,...,a_n)}.$$

□

Example 10.5. Let $V=\mathcal{Z}(x^2+y^2-1)\subseteq\mathbb{C}^2$. Then $\mathbb{C}(V)\cong\mathbb{C}(x,y)$ with $x^2+y^2=1$. Let $\varphi=\frac{1-y}{x}\in\mathbb{C}(V)$. Then φ is defined at $(0,1)\in V$ and $\varphi(0,1)=0$, but φ is not defined at $(0,-1)$.

Proof. Since $x^2=1-y^2$ in the ring $\mathbb{C}[V]$, we get

$$\varphi=\frac{1-y}{x}=\frac{(1-y)}{x}\cdot\frac{(1+y)}{(1+y)}=\frac{1-y^2}{x(1+y)}=\frac{x^2}{x(1+y)}=\frac{x}{1+y}$$

and $(1+y)(0,1)=1\neq 0$, we see that φ is defined at $(0,1)$ and $\varphi(0,1)=0$. On the other hand, suppose that φ is defined at $(0,-1)$, that is

$$\varphi=\frac{1-y}{x}=\frac{f}{g}$$

for some $f=F+\mathcal{I}(V)$, $g=G+\mathcal{I}(V)\in\mathbb{C}[V]$ with $G(0,-1)\neq 0$. Then $(1-y)G(x,y)=xF(x,y)$ in $\mathbb{C}[V]$. But this implies $1\cdot G(0,-1)=0\cdot F(0,-1)=0$, so that $G(0,-1)=0$ rendering such a presentation impossible. □

Remark 10.6. Let $V\subseteq k^n$ be an affine algebraic variety. Every rational function $\frac{f}{g}\in k(V)$, $f=F+\mathcal{I}(V)$, $g=G+\mathcal{I}(V)\in k[V]$, $F,G\in k[x_1,...,x_n]$ determines a function defined on some nonempty open subset $U\subseteq V$ with values in k that we shall also call a rational function.

Proof. Indeed, the set

$$\begin{aligned} U &= \{(a_1, ..., a_n) \in V \mid G(a_1, ..., a_n) \neq 0\} \\ &= V \setminus \{(a_1, ..., a_n) \in V \mid G(a_1, ..., a_n) = 0\} \\ &= V \setminus (V \cap \mathcal{Z}(G)) \end{aligned}$$

is open in the Zariski topology on V induced from k^n. To see that it is nonempty, suppose that $G(a_1, ..., a_n) = 0$ for all $(a_1, ..., a_n) \in V$. But then $G \in \mathcal{I}(V)$, that is $g = 0$ as an element of the coordinate ring $k[V]$, and thus g cannot be a denominator of a quotient in the field of fractions of $k[V]$. □

Remark 10.7. Let $V \subseteq k^n$ be an affine algebraic variety. If the rational functions $\varphi_1, \varphi_2 \in k(V)$ have the same values on a certain nonempty open subset of $U \subseteq V$, then they are equal.

Proof. Say $\varphi_1 = \frac{f_1}{g_1}$ and $\varphi_2 = \frac{f_2}{g_2}$ with $f_1 = F_1 + \mathcal{I}(V)$, $f_2 = F_2 + \mathcal{I}(V)$, $g_1 = G_1 + \mathcal{I}(V)$, $g_2 = G_2 + \mathcal{I}(V) \in k[V]$. If $\varphi_1 = \varphi_2$ on an open subset $U \subseteq V$, then

$$\frac{F_1}{G_1} - \frac{F_2}{G_2} = \frac{F_1 G_2 - F_2 G_1}{G_1 G_2} = 0$$

on U, that is $F_1 G_2 - F_2 G_1 = 0$ on U as a restriction of a polynomial function $k^n \to k$ to V. Clearly $F_1 G_2 - F_2 G_1$ is a continuous function in the Zariski topology, and by Remark 5.5 the set U is dense in V, so that $F_1 G_2 - F_2 G_1 = 0$ on V leading to $\varphi_1 = \varphi_2$ on V. □

Theorem 10.8. *Let $V \subseteq k^n$ be an affine algebraic variety over an algebraically closed field k. If the rational function $\varphi \in k(V)$ is defined at every point of V, then $\varphi \in k[V]$.*

Proof. Since φ is defined at every point of V, then for each such point $\underline{a} \in V$ there exist $f_{\underline{a}} = F_{\underline{a}} + \mathcal{I}(V)$, $g_{\underline{a}} = G_{\underline{a}} + \mathcal{I}(V) \in k[V]$ such that $\varphi = \frac{f_{\underline{a}}}{g_{\underline{a}}}$ with $G_{\underline{a}}(\underline{a}) \neq 0$. Let $\mathfrak{a} = (\{G_{\underline{a}} \mid \underline{a} \in V\}) \lhd k[x_1, ..., x_n]$. Since $k[x_1, ..., x_n]$ is Noetherian, there exists a finite number of points $\underline{a_1}, ..., \underline{a_m} \in V$ such that $\mathfrak{a} = (G_{\underline{a}_1}, ..., G_{\underline{a}_m})$. The polynomials $G_{\underline{a}_1}, ..., G_{\underline{a}_m}$ have no common zero on V, for if $G_{\underline{a}_1}(\underline{a}) = ... = G_{\underline{a}_m}(\underline{a}) = 0$ for some $\underline{a} \in V$, then $G_{\underline{a}}(\underline{a}) \neq 0$ and, as $G_{\underline{a}} \in \mathfrak{a}$, $G_{\underline{a}}(\underline{a}) = P_1(\underline{a}) G_{\underline{a}_1}(\underline{a}) + ... + P_m(\underline{a}) G_{\underline{a}_m}(a)$, for some $P_1, ..., P_m \in k[x_1, ..., x_n]$, so that $G_{\underline{a}}(\underline{a}) = 0$ – a contradiction. Therefore $\mathcal{Z}(\mathfrak{a} + \mathcal{I}(V)) = \emptyset$, and by Lemma 5.16 there exist $H_1, ..., H_m \in k[x_1, ..., x_n]$ and $Q \in \mathcal{I}(V)$ such that the following equation holds true in the ring $k[x_1, ..., x_n]$:

$$H_1 G_{\underline{a}_1} + ... + H_m G_{\underline{a}_m} + Q = 1.$$

But this leads to

$$\begin{aligned} (H_1 + \mathcal{I}(V))(G_{\underline{a}_1} + \mathcal{I}(V)) + ... + (H_m + \mathcal{I}(V))(G_{\underline{a}_m} + \mathcal{I}(V)) + (Q + \mathcal{I}(V)) = \\ &= (H_1 + \mathcal{I}(V))(G_{\underline{a}_1} + \mathcal{I}(V)) + ... + (H_m + \mathcal{I}(V))(G_{\underline{a}_m} + \mathcal{I}(V)) \\ &= 1 + \mathcal{I}(V) \end{aligned}$$

holding true in $k[V]$ and, consequently, $k(V)$. Multiplying both sides by φ and using the fact that $\varphi = \frac{f_{\underline{a}_i}}{g_{\underline{a}_i}}$, $i \in \{1, ..., m\}$, yields:

$$(H_1 + \mathcal{I}(V)) f_{\underline{a}_1} + ... + (H_m + \mathcal{I}(V)) f_{\underline{a}_m} = \varphi,$$

that is $\varphi \in k[V]$. □

Definition 10.9. *Let $V \subseteq k^n$ be an affine algebraic set, let $V = V_1 \cup \ldots \cup V_m$ be the decomposition of V into affine algebraic varieties. The **k-algebra of rational functions** of V is defined to be*

$$k(V) = k(V_1) \oplus \ldots \oplus k(V_m)$$

*and its elements are called **rational functions** on V.*

Definition 10.10. *Let $V \subseteq k^n$ be an affine algebraic set. If a rational function $\varphi \in k(V)$ is defined at every point of an open subset $U \subseteq V$, then the restriction $\varphi \upharpoonright_U$ wil be called a **regular** function on U.*

Example 10.11. Let $V = \mathcal{Z}(xy)$. Then $V = \mathcal{Z}(x) \cup \mathcal{Z}(y)$. Let $f = x(y+1)$. Then $f \upharpoonright_{\mathcal{Z}(x) \setminus \{(0,0)\}} = 0$ and $f \upharpoonright_{\mathcal{Z}(y) \setminus \{(0,0)\}} = 1$, $f \in k(V)$, f is regular on both $\mathcal{Z}(x)$ and $\mathcal{Z}(y)$, but not regular on V, as it is not defined on $(0,0)$.

Remark 10.12. Let $V \subseteq k^n$ be an affine algebraic set, let $f \in k(V)$. Then f is continuous on the set of points where it is defined.

Proof. It suffices to check that counterimages of closed sets are closed, which follows directly from the definition of Zariski topology. □

Theorem 10.13. *Let $V \subseteq k^n$ be an affine algebraic variety, let $f = F + \mathcal{I}(V) \in k[V] \setminus \{0\}$, $F \in k[x_1, \ldots, x_n]$, let*

$$k[V]_f = \left\{ \varphi \in k(V) | \, \varphi = \frac{h}{f}, m \in \mathbb{Z}, h \in k[V] \right\}$$

and

$$V_f = \{(a_1, \ldots, a_n) \in V | \, F(a_1, \ldots, a_n) \neq 0\}.$$

Then the k-algebra of regular functions on V_f is isomorphic to $k[V]_f$.

Proof. That every rational function from $k[V]_f$ is defined at every point of V_f and thus yields a regular function there – is clear.

Conversely, consider a rational function $\varphi \in k(V)$ regular on V_f. Following the proof of Theorem 10.8, for every $\underline{a} \in V_f$ there exist $h_{\underline{a}} = H_{\underline{a}} + \mathcal{I}(V)$, $f_{\underline{a}} = F_{\underline{a}} + \mathcal{I}(V) \in k[V]$ such that $\varphi = \frac{h_{\underline{a}}}{f_{\underline{a}}}$ with $F_{\underline{a}}(\underline{a}) \neq 0$. Let $\mathfrak{a} = (\{F_{\underline{a}} | \underline{a} \in V_f\}) \lhd k[x_1, \ldots, x_n]$. Then $\mathfrak{a} = (F_{\underline{a}_1}, \ldots, F_{\underline{a}_m})$, for some points $\underline{a_1}, \ldots, \underline{a_m} \in V_f$, and the polynomials $F_{\underline{a}_1}$, ..., $F_{\underline{a}_m}$ have no common zeros on V_f i.e. conceivable common zeros of $F_{\underline{a}_1}$, ..., $F_{\underline{a}_m}$ are among zeros of F. Thus $\mathcal{Z}(\mathfrak{a}) \subseteq \mathcal{Z}(F)$, hence $\mathfrak{a} \supseteq (F)$ and there exist G_1, ..., $G_m \in k[x_1, \ldots, x_n]$ such that

$$G_1 F_{\underline{a}_1} + \ldots + G_m F_{\underline{a}_m} = F$$

yielding

$$(G_1 + \mathcal{I}(V)) f_{\underline{a}_1} + \ldots + (G_m + \mathcal{I}(V)) f_{\underline{a}_m} = f,$$

which, after multiplying by φ and using $\varphi = \frac{h_{\underline{a}}}{f_{\underline{a}}}$, $\underline{a} \in V_f$, gives

$$(G_1 + \mathcal{I}(V)) h_{\underline{a}_1} + \ldots + (G_m + \mathcal{I}(V)) h_{\underline{a}_m} = f\varphi,$$

or, denoting by $h = (G_1 + \mathcal{I}(V)) h_{\underline{a}_1} + \ldots + (G_m + \mathcal{I}(V)) h_{\underline{a}_m} \in k[V]$:

$$h = f\varphi,$$

or, equivalently, $\varphi = \frac{h}{f}$. $\square$

Definition 10.14. *Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes. A* ***rational map*** *$f: V \to W$ is a map such that there exist $f_1, \ldots, f_m \in k(V)$ such that $f(a) = (f_1(a), \ldots, f_m(a))$, for all the points $a \in V$ where all the rational functions $f_1, \ldots, f_m \in k(V)$ are defined.*

Remark 10.15. Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes, let $f: V \to W$ be a rational map, $f = (f_1, \ldots, f_m)$ with $f_1, \ldots, f_m \in k(V)$. There exists an open set $\emptyset \neq U \subseteq V$ such that $f_1 \restriction_U, \ldots, f_m \restriction_U$ are regular on U. In other words, we can think of rational maps as defined on open subsets.

Proof. By Remark 10.6 there exist nonempty open subsets $U_1, \ldots, U_m \subseteq V$ such that $f_i \restriction_{U_i}$ is regular on U_i. The set $U = U_1 \cap \ldots \cap U_m$ is open, as a finite intersection of open sets, and it suffices to show that it is nonempty.

Indeed, suppose that $\bigcap_{i=1}^m U_i = \emptyset$. Say $U_i = V \setminus V_i$, for some closed subset V_i, $i \in \{1, \ldots, m\}$. But then $V_i \neq V$ and $V = \bigcup_{i=1}^m V_i$ contradicting the fact that V, as a variety, is irreducible. $\square$

Remark 10.16. Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes, let $f_1, \ldots, f_m \in k(V)$. Then $f_1, \ldots, f_m$ define a rational map $f: V \to W$.

Proof. It suffices to check that at every point $a \in V$ where all the rational functions $f_1, \ldots, f_m$ are defined we have, in fact, $(f_1(a), \ldots, f_m(a)) \in W$. Let $U \subseteq V$ be the nonempty open set such that $f_1 \restriction_U, \ldots, f_m \restriction_U$ are regular on U. Let $u \in \mathcal{I}(W)$. Then $u(f_1, \ldots, f_m) \in k(V)$ and $u(f_1, \ldots, f_m)$ vanishes at every point of U. As a nonempty open set in V, U is dense in V by Remark 5.5. $u(f_1, \ldots, f_m)$ is continuous and vanishes on the dense set U, so it vanishes on V. Since $u \in \mathcal{I}(W)$ was chosen arbitrarily, this yields $(f_1(a), \ldots, f_m(a)) \in W$. $\square$

Remark 10.17. Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes, let $f: V \to W$ be a rational map and assume that $f(V)$ is dense in W. The map f defines a field embedding $f^*: k(W) \to k(V)$.

Proof. Let $f_1, \ldots, f_m \in k(V)$ be such that $f = (f_1, \ldots, f_m)$ and let $U \subseteq V$ be the nonempty open set such that $f_1 \restriction_U, \ldots, f_m \restriction_U$ are regular on U. Consider f as a map $f: U \to f(V)$. For $\varphi \in k[W]$, $\varphi = \Phi + \mathcal{I}(W)$, $\Phi \in k[x_1, \ldots, x_m]$, define $f^*(\varphi) = \Phi(f_1, \ldots, f_m)$. Clearly $\Phi(f_1, \ldots, f_m) \in k[V] \subseteq k(V)$, so that $f^*: k[W] \to k(V)$ is a homomorphism, and it suffices to check that it is injective.

If $f^*(\varphi) = 0$ for $\varphi = \Phi + \mathcal{I}(W) \in k[W]$, then $\Phi = 0$ on $f(V)$. But if $\Phi \neq 0$ on W, then the equality $\Phi = 0$ defines a closed subset W' of W. Then $\varphi(V) \subseteq W'$, but this contradicts the assumption that $f(V)$ is dense in W.

The embedding $\varphi^*: k[W] \to k(V)$ can be extended in an obvious way to $\varphi^*: k(W) \to k(V)$. $\square$

Definition 10.18. *Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes, let $f: V \to W$ be a rational map such that $f(V)$ is dense in W. The map f is a* ***birational equivalence*** *if there is a rational map $g: W \to V$ such that $g(W)$ is dense in V and*

$$f \circ g = 1_W \text{ and } g \circ f = 1_V.$$

In this case we say that V and W are ***birationally equivalent*** *or* ***birational****.*

Corollary 10.19. *Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes. Then V and W are birationally equivalent if and only if $k(V) \cong k(W)$.*

Proof. If V and W are birationally equivalent, then $k(V) \cong k(W)$ by Remark 10.17. Conversely, if $k(V) \cong k(W)$, then the rational functions $x_i = X_i + \mathcal{I}(V)$ correspond to rational functions $f_i \in k(W)$. One checks that $f = (f_1, \ldots, f_m)$ is a birational equivalence. $\square$

Example 10.20. Let $V = \mathcal{Z}(xy - 1)$ and $W = \mathcal{Z}(y)$, let $f: V \to W$ be given by $(x, y) \mapsto (x, 0)$. This is a birational equivalence, but not an isomorphism.

Example 10.21. Let $V = \mathcal{Z}(y)$ and $W = \mathcal{Z}(y^2 - x^3)$, let $f: V \to W$ be given by $(x, 0) \mapsto (x^2, x^3)$. This is a birational equivalence (the inverse map $g: W \to V$ being $(x, y) \mapsto \left(\frac{y}{x}, 0\right)$), but not an isomorphism.

Proposition 10.22. *Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes, let $f: V \to W$ be a birational equivalence. Then there exist open subsets $U \subseteq V$ and $U' \subseteq W$ which are isomorphic.*

Proof. Let $g: W \to V$ be the birational map such that $f \circ g = 1_W$ and $g \circ f = 1_V$. Let $U_1 \subseteq V$ be the open set on which f is defined, and, likewise, $U_2 \subseteq W$ the open set where g is defined. Then $f \circ g$ is the identity map on $U_2 \cap g^{-1}(U_1)$ and $g \circ f$ is the identity map on $U_1 \cap f^{-1}(U_2)$. Thus f and g define an isomorphism between $U = f^{-1} \circ g^{-1}(U_1)$ and $U' = g^{-1} \circ f^{-1}(U_2)$. □

Proposition 10.23. (Noether normalization lemma) *Let k be algebraically closed, and $k \subseteq K$ a finitely generated field extension. Then there exist elements $z_1, \ldots, z_{d+1} \in K$ with $K = k(z_1, \ldots, z_{d+1})$ such that $z_1, \ldots, z_d$ are algebraically independent over k, and z_{d+1} is separable over $k(z_1, \ldots, z_{d+1})$.*

Proof. Let K be generated over k by a finite number of elements $t_1, \ldots, t_n$ and let d be the maximal number of algebraically independent elements among $t_1, \ldots, t_n$. Changing the order of $t_1, \ldots, t_n$, if necessary, we might as well assume that $t_1, \ldots, t_d$ are algebraically independent. Then any element $y \in K$ is algebraically dependent on $t_1, \ldots, t_d$ and, moreover, there exists a relation $f(t_1, \ldots, t_d, y) = 0$ with $f(T_1, \ldots, T_d, T_{d+1})$ irreducible ober k.

Let $f(T_1, \ldots, T_d, T_{d+1})$ be such a polynomial for $t_1, \ldots, t_d, t_{d+1}$. We claim that the partial derivative $\frac{\partial f}{\partial T_i}(T_1, \ldots, T_d, T_{d+1}) \neq 0$ for at least one $i \in \{1, \ldots, d+1\}$. Indeed, if this was not the case, then each T_i occurs in f in powers that are multiples of the characteristic p of the field k, that is, f is of the form $f = \sum a_{i_1 \ldots i_{d+1}} T_1^{p i_1} \cdot \ldots \cdot T_{d+1}^{p i_{d+1}}$. Set $a_{i_1 \ldots i_{d+1}} = b_{i_1 \ldots i_{d+1}}^p$ and $g = f = \sum b_{i_1 \ldots i_{d+1}} T_1^{i_1} \cdot \ldots \cdot T_{d+1}^{i_{d+1}}$. Then we get $f = g^p$, which contradicts the irreducibility of f.

If $\frac{\partial f}{\partial T_i}(T_1, \ldots, T_d, T_{d+1}) \neq 0$, the d elements $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{d+1}$ are algebraically independent over k. Indeed, t_i is algebraically independent over $k(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{d+1})$ because $\frac{\partial f}{\partial T_i}(T_1, \ldots, T_d, T_{d+1}) \neq 0$, so that T_i occurs in f. Thus if $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{d+1}$ were algebraically dependent, the transcendence degree of $k(t_1, \ldots, t_{d+1})$ would be less than d, which contradicts the algebraic independence of $t_1, \ldots, t_d$.

Thus we can always rearrange $t_1, \ldots, t_{d+1}$ so that $t_1, \ldots, t_d$ are algebraically independent over k, and $\frac{\partial f}{\partial T_{d+1}}(T_1, \ldots, T_d, T_{d+1}) \neq 0$. This shows that t_{d+1} is separable over $k(t_1, \ldots, t_d)$. Since t_{d+2} is algebraic over $k(t_1, \ldots, t_d)$, by the Primitive Element Theorem we can find an element $y \in K$ such that $k(t_1, \ldots, t_{d+2}) = k(t_1, \ldots, t_d, y)$. Repeating the process of adjoining elements $t_{d+1}, \ldots, t_n$ we express K as $k(z_1, \ldots, z_{d+1})$, where $z_1, \ldots, z_d$ are algebraically independent over k and $f(z_1, \ldots, z_d, z_{d+1}) = 0$, with f an irreducible polynomial over k with $\frac{\partial f}{\partial T_{d+1}}(T_1, \ldots, T_d, T_{d+1}) \neq 0$. □

Proposition 10.24. *Let $V \subseteq k^n$ be an affine variety. Then V is birationally equivalent to a hypersurface of some affine space k^m.*

Proof. $k(V)$ is finitely generated over, say $k(V) = k(t_1, \ldots, t_n)$. We may view $t_1, \ldots, t_n$ as rational functions on V. Let d be the maximal number of $t_1, \ldots, t_d$ that are algebraically independent over k. By the Noether Normalization Lemma, $k(V)$ can be written in the form $k(z_1, \ldots, z_{d+1})$, where $z_1, \ldots, z_d$ are algebraically independent and $f(z_1, \ldots, z_{d+1}) = 0$ for some irreducible polynomial $f \in k[T_1, \ldots, T_{d+1}]$ with $\frac{\partial f}{\partial T_{d+1}}(T_1, \ldots, T_d, T_{d+1}) \neq 0$. Let $W = \mathcal{Z}(f)$. The function field $k(W)$ of the variety W is obviously isomorphic to $k(V)$, which means that V and W are birationally equivalent. □

A variety is called **rational** if it is birationally equivalent to k^n, for some n.