

9 Coordinate ring of an affine algebraic set. Morphisms of affine algebraic sets. Category of affine algebraic sets.

9.1 Coordinate ring of an affine algebraic set.

Definition 9.1. Let k be a field, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V . The ring $k[V] := k[x_1, \dots, x_n]/\mathcal{I}(V)$ is called the **coordinate ring** of V .

Remark 9.2. Let k be a field, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V . Let $f \in k[x_1, \dots, x_n]$. The polynomial f defines a polynomial function $k^n \rightarrow k$. Let f_V be the restriction of f to the set V , $f_V = f|_V$. Then $f_V = g_V$ if and only if $f + \mathcal{I}(V) = g + \mathcal{I}(V)$.

Proof. Indeed, $f_V = g_V$ means that $f(a_1, \dots, a_n) = g(a_1, \dots, a_n)$, for all $(a_1, \dots, a_n) \in V$, that is $(f - g)(a_1, \dots, a_n) = 0$, for all $(a_1, \dots, a_n) \in V$, or, equivalently, $f - g \in \mathcal{I}(V)$. \square

Remark 9.3. Let k be a field, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V . Let $\kappa: k[x_1, \dots, x_n] \rightarrow k[V]$ be the canonical epimorphism, $\kappa(f) = \bar{f} := f + \mathcal{I}(V)$. Then $k[V]$ is a k -ring finitely generated over k by $\bar{x}_1, \dots, \bar{x}_n$.

Remark 9.4. Let k be algebraically closed, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V . Then $k[V]$ has no nonzero nilpotents.

Proof. By Hilbert Nullstellensatz, $\mathcal{I}(V)$ is radical, so that, by Lemma 6.7, $k[V] = k[x_1, \dots, x_n]/\mathcal{I}(V)$ has no nonzero nilpotents. \square

Theorem 9.5. Let k be algebraically closed. Then a k -ring A is isomorphic to a coordinate ring of an affine algebraic set $V \subseteq k^n$ if and only if it is finitely generated over k and has no nonzero nilpotents.

Proof. Let $A = k[t_1, \dots, t_n]$ be a ring finitely generated over k with no nonzero nilpotents. The map

$$k[x_1, \dots, x_n] \rightarrow A, \quad f \mapsto f(t_1, \dots, t_n)$$

is a well-defined ring epimorphism. Define by \mathfrak{a} its kernel. The ring $k[x_1, \dots, x_n]/\mathfrak{a} \cong A$ has no nonzero nilpotents, hence, by Lemma 6.7, the ideal \mathfrak{a} is radical. Thus $\mathfrak{a} = \mathcal{I}(\mathcal{Z}(\mathfrak{a}))$ and, consequently, $A \cong k[\mathcal{Z}(\mathfrak{a})]$. \square

Example 9.6. One easily checks that:

- $V = k^n$, $k[V] \cong k[x_1, \dots, x_n]$;
- $V = \emptyset$, $k[V] \cong 0$;
- $V = \{(a_1, \dots, a_n)\}$, $k[V] \cong k$.

Example 9.7. Let $V = \mathcal{Z}(f)$, where $f \in k[x_1, \dots, x_n]$ is square-free and k is algebraically closed. Then $k[V] \cong k[x_1, \dots, x_n]/(f) \cong k[\alpha_1, \dots, \alpha_n]$ where $f(\alpha_1, \dots, \alpha_n) = 0$.

Proof. By Hilbert Nullstellensatz $\mathcal{I}(V) = \mathcal{I}(\mathcal{Z}(f)) = \text{rad}(f)$. One easily checks that $(f) = \text{rad}(f)$ if and only if f is square-free, which follows that $k[V] \cong k[x_1, \dots, x_n]/(f) \cong k[\alpha_1, \dots, \alpha_n]$, where $\alpha_i = x_i + (f)$, for $i \in \{1, \dots, n\}$. \square

Example 9.8. Let $V = \mathcal{Z}(a_1x_1 + \dots + a_nx_n - b)$, where $a_1, \dots, a_n, b \in k$ and k is algebraically closed. Then $k[V] \cong k[x_1, \dots, x_{n-1}]$.

Proof. As in the previous example, $k[V] \cong k[x_1, \dots, x_n] / (a_1x_1 + \dots + a_nx_n - b) \cong k[\alpha_1, \dots, \alpha_n]$, where $a_1\alpha_1 + \dots + a_n\alpha_n = b$. Relabelling, if necessary, we may assume that $a_n \neq 0$. Further, we may assume that $a_n = 1$, since $\mathcal{Z}(a_1x_1 + \dots + a_nx_n - b) = \mathcal{Z}\left(\frac{a_1}{a_n}x_1 + \dots + \frac{a_n}{a_n}x_n - \frac{b}{a_n}\right)$. Thus

$$\alpha_n = b - a_1\alpha_1 - \dots - a_{n-1}\alpha_{n-1},$$

and the ring $k[V] \cong k[\alpha_1, \dots, \alpha_n]$ is generated by the elements $\alpha_1, \dots, \alpha_{n-1}$. It suffices to show that these elements are algebraically independent: indeed, if $g(\alpha_1, \dots, \alpha_{n-1}) = 0$, for some $g \in k[x_1, \dots, x_{n-1}]$, then

$$\begin{aligned} \mathcal{I}(V) &= 0_{k[V]} = g(\alpha_1, \dots, \alpha_{n-1}) = g(x_1 + \mathcal{I}(V), \dots, x_{n-1} + \mathcal{I}(V)) \\ &= g(x_1, \dots, x_{n-1}) + \mathcal{I}(V), \end{aligned}$$

so that $g \in \mathcal{I}(V) = (h)$, that is h divides g in the ring $k[x_1, \dots, x_n]$. But this is impossible, since x_n appears in h with a nonzero coefficient, and does not appear in any of the monomials of g . Thus $\alpha_1, \dots, \alpha_{n-1}$ are algebraically independent over k , and thus $k[\alpha_1, \dots, \alpha_{n-1}] \cong k[x_1, \dots, x_{n-1}]$. \square

9.2 Category of affine algebraic sets.

Definition 9.9. Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine algebraic sets. A **morphism** $f: V \rightarrow W$ is a map such that there exist $f_1, \dots, f_m \in k[V]$ such that $f(a) = (f_1(a), \dots, f_m(a))$, for all $a \in V$.

Remark 9.10. Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine algebraic sets, let $f_1, \dots, f_m \in k[V]$. Then $f = (f_1, \dots, f_m): V \rightarrow W$ is a morphism if and only if

$$g(f_1, \dots, f_m) = 0 \in k[V] \quad \text{for all } g \in \mathcal{I}(W).$$

Proof. Indeed, one easily checks that

$$\begin{aligned} (f_1(a_1, \dots, a_n), \dots, f_m(a_1, \dots, a_n)) \in W &\Leftrightarrow g(f_1(a_1, \dots, a_n), \dots, f_m(a_1, \dots, a_n)) = 0 \text{ for all } g \in \mathcal{I}(W) \\ &\Leftrightarrow g(f_1, \dots, f_m)(a_1, \dots, a_n) = 0 \text{ for all } g \in \mathcal{I}(W) \\ &\Leftrightarrow g(f_1, \dots, f_m) \in \mathcal{I}(V) \text{ for all } g \in \mathcal{I}(W) \\ &\Leftrightarrow g(f_1, \dots, f_m) = 0 \in k[V] \text{ for all } g \in \mathcal{I}(W). \end{aligned}$$

\square

Example 9.11. Consider the following easy examples.

- Let $f \in k[V]$. Then $f: V \rightarrow k$ is a morphism.
- Let $f: k^n \rightarrow k^m$ be a linear map. Then f is a morphism.
- Let $f: \mathcal{Z}(xy - 1) \rightarrow k$ be given by $f(x, y) = x$. Then f is a morphism.
- Let $f: k \rightarrow \mathcal{Z}(y^2 - x^3)$ be given by $f(t) = (t^2, t^3)$. Then f is a morphism.

Example 9.12. One easily checks that:

- $\mathcal{Z}(y - x^k) \cong k$ via $f(x, y) = x$ and $g(t) = (t, t^k)$;
- $f: \mathcal{Z}(xy - 1) \rightarrow k$ given by $f(x, y) = x$ is not an isomorphism;
- $f: k \rightarrow \mathcal{Z}(y^2 - x^3)$ given by $f(t) = (t^2, t^3)$ is not an isomorphism, even though it is a bijection.

* We shall denote $k - \text{Aff}$ the category of affine algebraic sets over an algebraically closed field k with morphisms defined above.

Theorem 9.13. *Let k be algebraically closed and consider the categories $k - \text{Aff}$ and $k - \text{Alg}_{\text{fg}}^0$. The assignment*

$$F(V) = k[V] \text{ for an affine algebraic set } V \subseteq k^n$$

and

$$F(\varphi) = \varphi^* \text{ for a morphism of affine algebraic sets } \varphi: V \rightarrow W,$$

where $\varphi^*: k[W] \rightarrow k[V]$ is given by the formula

$$\varphi^*(f) = g \circ \varphi$$

defines an equivalence of categories $k - \text{Aff}^{\text{op}}$ and $k - \text{Alg}_{\text{fg}}^0$.

Proof. The map F assigns to an affine algebraic set $V \subseteq k^n$ a finitely generated k -algebra with no nilpotent elements by Theorem 9.5. By the same result F is also essentially surjective. If $\varphi: V \rightarrow W$ is a morphism between affine algebraic sets V and W , and if $g \in k[W]$, then $g \circ \varphi \in k[V]$. Moreover, the map $\varphi^*: k[W] \rightarrow k[V]$ is a homomorphism of k -algebras. Thus F defines a contravariant functor between categories $k - \text{Aff}$ and $k - \text{Alg}_{\text{fg}}^0$, or, equivalently, a covariant functor between the categories $k - \text{Aff}^{\text{op}}$ and $k - \text{Alg}_{\text{fg}}^0$. It remains to check that it is fully faithful.

Assume that $\varphi, \psi: V \rightarrow W$ are morphisms of affine algebraic sets and that $\varphi^* = \psi^*$. Say $\varphi = (\varphi_1, \dots, \varphi_m)$ and $\psi = (\psi_1, \dots, \psi_m)$ with $\varphi_1, \dots, \varphi_m, \psi_1, \dots, \psi_m \in k[V]$. Consider the element $\bar{x}_1 = x_1 + \mathcal{I}(V) \in k[W]$. Since $\bar{x}_1 \circ \varphi = \varphi^*(\bar{x}_1) = \psi^*(\bar{x}_1) = \bar{x}_1 \circ \psi$ it follows that $\varphi_1 = x_1(\varphi_1, \dots, \varphi_m) = x_1(\psi_1, \dots, \psi_m) = \psi_1$. Likewise $\varphi_j = \psi_j$, for $j \in \{2, \dots, m\}$, so that the map $\text{Hom}_{k - \text{Aff}}(V, W) \rightarrow \text{Hom}_{k - \text{Alg}_{\text{fg}}^0}(F(V), F(W))$ is injective.

Finally, let $f: k[W] \rightarrow k[V]$ be a homomorphism of k -algebras. We shall show that $f = \varphi^*$, for some morphism $\varphi: V \rightarrow W$. Indeed, consider the elements $\bar{x}_j = x_j + \mathcal{I}(W)$, for $j \in \{1, \dots, m\}$. Then $\varphi_j = f(\bar{x}_j) \in k[V]$, $j \in \{1, \dots, m\}$, and consider the map $\varphi = (\varphi_1, \dots, \varphi_m): V \rightarrow k^m$. Clearly $f = \varphi^*$ and all that is left to show is that $\varphi(V) \subseteq W$. Fix $H \in \mathcal{I}(W)$. Then $H(\bar{x}_1, \dots, \bar{x}_m) = 0$ in $k[W]$, hence also $f(H) = 0$ on V . Fix $(a_1, \dots, a_n) \in V$; then $H(\varphi(a_1, \dots, a_n)) = f(H)(a_1, \dots, a_n) = 0$, and therefore $\varphi(a_1, \dots, a_n) \in W$. \square