8 Morphisms and functors.

Remark 8.1. Let $\mathcal{C}$ be a category, let 0 be a zero object. Then for every pair of objects C, $D \in Ob(\mathcal{C})$ there exists exactly one morphism $C \xrightarrow{0_{C,D}} D$ such that

$$f \circ 0_{C,D} = 0_{C,E} \text{ and } 0_{C,D} \circ g = 0_{B,D},$$

for all morphisms $h \in \mathrm{Hom}(D, E)$ and $g \in \mathrm{Hom}(B, C)$. The morphism $0_{C,D}$ is called the **zero morphism.**

The proof is left to the reader as an excercise.

Definition 8.2. *Let $\mathcal{C}$ be a category, let $C, D \in Ob(\mathcal{C})$, let $C \overset{f}{\underset{g}{\rightrightarrows}} D$ be morphisms. An* ***equalizer*** *of the pair f, g is the pair (E, e) consisting of an object E and a morphism $E \xrightarrow{e} C$ such that*

1. *$f \circ e = g \circ e$;*

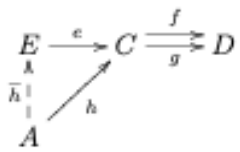
2. *if A is any object and $A \xrightarrow{h} C$ a morphism such that*

$$f \circ h = g \circ h,$$

then there exists exactly one morphism $A \xrightarrow{\bar{h}} E$ such that

$$e \circ \bar{h} = h.$$

In other words the diagram



is commutative.

Let $\mathcal{C}$ be a category, let $C, D \in Ob(\mathcal{C})$, let $C \overset{f}{\underset{g}{\rightrightarrows}} D$ be morphisms. A ***coequalizer*** *of the pair f, g is the pair (Q, q) consisting of an object Q and a morphism $D \xrightarrow{q} Q$ such that*

1. *$q \circ f = q \circ g$;*

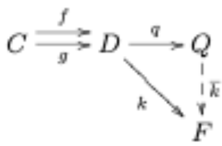
2. *if F is any object and $D \xrightarrow{k} F$ a morphism such that*

$$k \circ f = k \circ g,$$

then there exists exactly one morphism $Q \xrightarrow{\bar{k}} F$ such that

$$\bar{k} \circ q = k.$$

In other words the diagram



is commutative.

Example 8.3.

1. Consider the category $\mathcal{S}$et. Let $C, D \in \mathrm{Ob}(\mathcal{S}\mathrm{et})$, let $C \underset{g}{\overset{f}{\rightrightarrows}} D$ be morphisms. Define:

$$E = \{c \in C \colon f(c) = g(c)\}$$

and let $E \xrightarrow{e} C$ be the inclusion. Then (E, e) is an equalizer of the pair f, g.

2. Consider the category $\mathcal{G}$rp. Let $C, D \in \mathrm{Ob}\,(\mathcal{G}\mathrm{rp})$, let $C \underset{g}{\overset{f}{\rightrightarrows}} D$ be morphisms. Define:

$$E = \{c \in C \colon f(c) = g(c)\}$$

and let $E \xrightarrow{e} C$ be the inclusion. Then (E, e) is an equalizer of the pair f, g.

3. Consider the category $\mathcal{R}$ng. Let $C, D \in \mathrm{Ob}(\mathcal{R}\mathrm{ng})$, let $C \underset{g}{\overset{f}{\rightrightarrows}} D$ be morphisms. Define:

$$E = \{c \in C \colon f(c) = g(c)\}$$

and let $E \xrightarrow{e} C$ be the inclusion. Then (E, e) is an equalizer of the pair f, g.

4. Consider the category $R - \mathcal{M}$od. Let $C, D \in \mathrm{Ob}(R - \mathcal{M}\mathrm{od})$, let $C \underset{g}{\overset{f}{\rightrightarrows}} D$ be morphisms. Define:

$$E = \{c \in C \colon f(c) = g(c)\}$$

and let $E \xrightarrow{e} C$ be the inclusion. Then (E, e) is an equalizer of the pair f, g.

5. Consider the category $\mathcal{G}$rp. Let $C, D \in \mathrm{Ob}(\mathcal{G}\mathrm{rp})$, let $C \underset{g}{\overset{f}{\rightrightarrows}} D$ be morphisms. Define:

$$Q' = \text{the least normal subgroup of } D \text{that contains } \{f(c)\, g(c)^{-1} \colon c \in C\}$$

and let $D \xrightarrow{q} D / Q' = Q$ be the canonical epimorphism. Then (Q, q) is the coequalizer of the pair f, g.

Remark 8.4. Let $\mathcal{C}$ be a category, let $C, D \in Ob(\mathcal{C})$, let $C \underset{g}{\overset{f}{\rightrightarrows}} D$ be morphisms, let (E, e) be the equalizer, and (Q, q) a coequalizer of f, g. Then:

1. e is a monic,

2. q is an epic.

The proof is left to the reader as an excercise.

Definition 8.5. *Let $\mathcal{C}$ be a category, let 0 be the zero object, let $C, D \in Ob(\mathcal{C})$, let $C \xrightarrow{f} D$ be a morhism. A **kernel** of f is the equalizer (E, e) of the pair $f, 0_{C,D}$. The equalizer (E, e) is then denoted by* $(\operatorname{Ker} f, \ker f)$.

*A **cokernel** of f is the coequalizer (Q, q) of the pair $f, 0_{C,D}$. The coequalizer (Q, q) is then denoted by* $(\operatorname{Coker} f, \operatorname{coker} f)$.

Corollary 8.6. *Let $\mathcal{C}$ be a category, let 0 be the zero object, let $C, D \in Ob(\mathcal{C})$, let $C \xrightarrow{f} D$ be a morhism. Then*

1. $\ker f$ *is a monic and* $f \circ \ker f = 0_{\operatorname{Ker} f, D}$;

2. $\operatorname{coker} f$ *is an epic and* $\operatorname{coker} f \circ f = 0_{C, \operatorname{Coker} f}$.

The proof is left to the reader as an excercise.

Example 8.7.

1. Consider the category $\mathcal{G}$rp. Let $C, D \in Ob(\mathcal{G}\text{rp})$, let $C \xrightarrow{f} D$ be a morphism. A kernel of f is the pair $(\operatorname{Ker} f, \ker f)$, where

$$\operatorname{Ker} f = \{c \in C : f(c) = 0\}$$

   and $\operatorname{Ker} f \xrightarrow{\ker f} C$ is the inclusion map.

2. Consider the category $\mathcal{G}$rp. Let $C, D \in \operatorname{Ob}(\mathcal{G}\text{rp})$, let $C \xrightarrow{f} D$ be a morphism. A cokernel of f jis the pair $(\operatorname{Coker} f, \operatorname{coker} f)$, where

$$\operatorname{Coker} f = D / \operatorname{Im} f$$

   and $D \xrightarrow{\operatorname{coker} f} \operatorname{Coker} f$ is the canonical epimorphism.

Definition 8.8. *Let $\mathcal{C}$ be a category, let $A, B, Z \in Ob(\mathcal{C})$. Let $A \xrightarrow{f} Z$ and $B \xrightarrow{g} Z$ be morphisms. A **pullback** of the pair f, g is the triple (P, p, q) consisting of an object P and morphisms $P \xrightarrow{p} A$ and $P \xrightarrow{q} B$ such that:*

1. $f \circ p = g \circ q$;

2. *for every object Q and morphisms $Q \xrightarrow{r} A$ and $Q \xrightarrow{s} B$ such that $f \circ r = g \circ s$, there exists exactly one morphism $Q \xrightarrow{u} P$ such that $r = p \circ u$ and $s = q \circ u$;*

in other words the diagram

is commutative. The object P is denoted by $A \times_Z B$, the morphism p is called the ***pulback of g along f,*** *and q is called the* ***pulback of f along g.*** *Let $\mathcal{C}$ be a category, let $A, B, Z \in Ob(\mathcal{C})$. Let $Z \xrightarrow{f} A$ and $Z \xrightarrow{g} B$ be morphisms. A* ***pushout*** *of the pair f, g is the triple (P, p, q) consisting of an object P and morphisms $A \xrightarrow{p} P$ and $B \xrightarrow{q} P$ such that*

1. *$p \circ f = q \circ g$;*
2. *for every object Q and morphisms $A \xrightarrow{r} Q$ and $B \xrightarrow{s} Q$ such that $r \circ f = s \circ g$, there exists exactly one morphism $P \xrightarrow{u} Q$ such that $r = u \circ p$ and $s = u \circ q$;*

in other words the diagram

is commutative. The object P is denoted by $A \cup_Z B$, the morphism p is called the ***pushout of g along f,*** *and q is called the* ***pushout of f along g.***

Remark 8.9. Let $\mathcal{C}$ be a category. Then binary products and equalizers in $\mathcal{C}$ exist of and only if there exist pullbacks.

Proof. ($\Rightarrow$): Suppose that in a category $\mathcal{C}$ there exist binary products and equalizers. Fix $A, B, Z \in Ob(\mathcal{C})$ together with morphisms $A \xrightarrow{f} Z$ and $B \xrightarrow{g} Z$ and consider the product $A \times B$ of A and B together with canonical projections $A \times B \xrightarrow{\pi_1} A$ and $A \times B \xrightarrow{\pi_2} B$. Consider the diagram

$$A \times B \underset{g \circ \pi_2}{\overset{f \circ \pi_1}{\rightrightarrows}} Z.$$

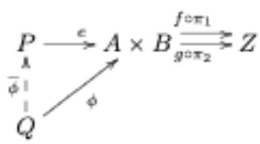
Let (P, e) be the equalizer of the pair $f \circ \pi_1$ and $g \circ \pi_2$:

$$P \xrightarrow{e} A \times B \underset{g \circ \pi_2}{\overset{f \circ \pi_1}{\rightrightarrows}} Z.$$

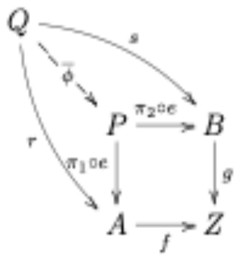
Then $(P, \pi_1 \circ e, \pi_2 \circ e)$ is the pullback of the pair f, g. Indeed, clearly $f \circ \pi_1 \circ e = g \circ \pi_2 \circ e$. Fix an object Q together with morphisms $Q \xrightarrow{r} A$ and $Q \xrightarrow{s} B$ such that $f \circ r = g \circ s$. By the universal property of the product there exists exactly one morphism $Q \xrightarrow{\phi} A \times B$ such that

$$\pi_1 \circ \phi = r \text{ and } \pi_2 \circ \phi = s.$$

By the universal property of the equalizer there exists exactly one morphism $Q \xrightarrow{\bar{\phi}} P$ such that the diagram



is commutative. Thus the diagram



is also commutative.

($\Leftarrow$): excercise. □

Remark 8.10. Let $\mathcal{C}$ be a category. Then binary coproducts and coequalizers in $\mathcal{C}$ exist of and only if there exist pushouts.

The proof is left to the reader as an excercise.

Example 8.11.

1. Consider the category $\mathcal{S}$et. Let $A, B, Z \in Ob(\mathcal{S}\text{et})$, let $A \xrightarrow{f} Z$ and $B \xrightarrow{g} Z$ be morphisms. The pullback of the pair f, g is the set:

$$A \times_Z B = \{(a, b) \in A \times B: f(a) = g(b)\}$$

together with morphisms $A \times_Z B \xrightarrow{p} A$ and $A \times_Z B \xrightarrow{q} B$ which are restrictions of the canonical projections to the set $A \times_Z B$.

Definition 8.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A covariant functor $F: \mathcal{C} \to \mathcal{D}$ is a pair of maps (F_1, F_2), $F_1: Ob(\mathcal{C}) \to Ob(\mathcal{D})$, $F_2: Ar(\mathcal{C}) \to Ar(\mathcal{D})$ such that*

1. *for $A \in Ob(\mathcal{C})$ and for $B = F_1(A) \in Ob(\mathcal{D})$*

$$F_2(1_A) = 1_B;$$

2. *for $A \xrightarrow{f} B$, $A, B \in Ob(\mathcal{C})$*

$$F_1(A) \xrightarrow{F_2(f)} F_1(B);$$

3. *for $B \xrightarrow{f} C$, $A \xrightarrow{g} B$, $A, B, C \in Ob(\mathcal{C})$:*

$$F_2(f \circ g) = F_2(f) \circ F_1(g)$$

Definition 8.13. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A contravariant functor $F: \mathcal{C} \to \mathcal{D}$ is a pair of maps (F_1, F_2), $F_1: Ob(\mathcal{C}) \to Ob(\mathcal{D})$, $F_2: Ar(\mathcal{C}) \to Ar(\mathcal{D})$ such that*

1. *for $A \in Ob(\mathcal{C})$ and for $B = F_1(A) \in Ob(\mathcal{D})$*

$$F_2(1_A) = 1_B;$$

2. *for $A \xrightarrow{f} B$, $A, B \in Ob(\mathcal{C})$*

$$F_1(B) \xrightarrow{F_2(f)} F_1(A);$$

3. *for* $B \xrightarrow{f} C$, $A \xrightarrow{g} B$, $A, B, C \in Ob(\mathcal{C})$:

$$F_2(f \circ g) = F_2(g) \circ F_1(f)$$

We shall routinely skip the indexes F_1, F_2 and simply write F istead for both maps.

Example 8.14.

1. For any category $\mathcal{C}$ the assignment $I_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$ given by:

$$I_{\mathcal{C}}(A) = A, A \in Ob(\mathcal{C}) \text{ and } I_{\mathcal{C}}(f) = f, f \in Ar(\mathcal{C})$$

   is a covariant functor that we shall call the **identity functor**.

2. The assignment $F: \mathcal{G}\text{rp} \to \mathcal{S}\text{et}$ given by:

$$F(A) = A, A \in Ob(\mathcal{G}\text{rp}) \text{ and } F(f) = f, f \in Ar(\mathcal{G}\text{rp})$$

   is a covariant functor that we shall call the **forgetful functor**. Likewise, we can define forgetful functors $\mathcal{R}\text{ng} \to \mathcal{S}\text{et}$, $R - \mathcal{M}\text{od} \to \mathcal{S}\text{et}$, $\mathcal{T}\text{op} \to \mathcal{S}\text{et}$, or, for example, $\mathcal{R}\text{ng} \to \mathcal{G}\text{rp}$.

3. The assignment $F: \mathcal{S}\text{et} \to \mathcal{A}b$ given by:

$$F(X) = \text{free abelian group with basis } X, X \in Ob(\mathcal{S}\text{et})$$

   and

$$F(f) = \text{unique extension of } f \text{to } \bar{f}: F(X_1) \to F(X_2), f \in Hom(X_1, X_2)$$

   is a covariant functor that creates free objects. Likewise, we can define functors creating free objects $\mathcal{S}\text{et} \to \mathcal{G}\text{rp}$, or $\mathcal{S}\text{et} \to R - \mathcal{M}\text{od}$.

4. For any category $\mathcal{C}$ and $A \in Ob(\mathcal{C})$ the assignment $h_A: \mathcal{C} \to \mathcal{S}\text{et}$ given by:

$$h_A(C) = \text{Hom}(A, C), C \in Ob(\mathcal{C})$$

   and

$$h_A(f) = \bar{f}, f \in Ar(\mathcal{C}),$$

   where $C \xrightarrow{f} C'$ and $\bar{f}: \text{Hom}(A, C) \to \text{Hom}(A, C')$ is given by

$$\bar{f}(\phi) = f \circ \phi$$

   is a covariant functor that we shall call the **covariant hom functor**.

5. For any category $\mathcal{C}$ and $A \in Ob(\mathcal{C})$ the assignment $h_A: \mathcal{C} \to \mathcal{S}\text{et}$ given by:

$$h_A(C) = \text{Hom}(C, A), C \in Ob(\mathcal{C})$$

   and

$$h_A(f) = \bar{f}, f \in Ar(\mathcal{C}),$$

where $C \xrightarrow{f} C'$ and $\bar{f}\colon \mathrm{Hom}(C', A) \to \mathrm{Hom}(C, A)$ is given by

$$\bar{f}(\psi) = \psi \circ f$$

is a contravariant functor that we shall call the **contravariant hom functor**.

6. For a category $\mathcal{C}$ denote by $\mathcal{C}^{op}$ the **opposite category** defined as follows: $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$ and, for $A, B \in Ob(\mathcal{C}^{op})$:

$$\mathrm{Hom}_{\mathcal{C}^{op}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(B, A),$$

and

$$f^{op} \circ g^{op} = (g \circ f)^{op}.$$

For example, if $\mathcal{C}$ is the following category:

$$A \to B \to C \to D$$

then $\mathcal{C}^{op}$ is:

$$A \leftarrow B \leftarrow C \leftarrow D.$$

If $F\colon \mathcal{C} \to \mathcal{D}$ is a contravariant functor, then $\bar{F}\colon \mathcal{C}^{op} \to \mathcal{D}$ defined by

$$\bar{F}(A) = F(A), A \in Ob(\mathcal{C}), \text{and } \bar{F}(f^{op}) = F(f), f \in Ar(\mathcal{C}^{op}),$$

is a covariant functor.

7. The asignment $F\colon R - \mathcal{M}\mathrm{od} \to R - \mathcal{M}\mathrm{od}$ given by:

$$F(A) = \mathrm{Hom}(A, R), A \in Ob(R - \mathcal{M}\mathrm{od}) \text{ and } F(f) = f^*, f \in Ar(R - \mathcal{M}\mathrm{od}),$$

where $A \xrightarrow{f} A'$ and $f^*\colon \mathrm{Hom}(A, R) \to \mathrm{Hom}(A', R)$ is given by

$$f^*(\phi) = \phi \circ f$$

is a contravariant functor that is a special case of the contravariant hom functor. It is usually denoted by * and called the **dual functor**. In particular, when $R = F$ is a field and $R - \mathcal{M}\mathrm{od}$ becomes the category of F-vector spaces, this yields the familiar construction of the dual vector space.

8. The asignment $^{**}\colon R - \mathcal{M}\mathrm{od} \to R - \mathcal{M}\mathrm{od}$ given by:

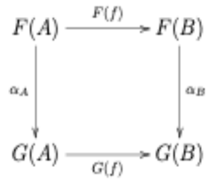
$$A^{**} = (A^*)^*, A \in Ob(R - \mathcal{M}\mathrm{od}) \text{ and } f^{**} = (f^*)^*, f \in Ar(R - \mathcal{M}\mathrm{od}),$$

is a covariant functor that called the **double dual** (or **bidual**) **functor**.

Definition 8.15. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, let $F, G\colon \mathcal{C} \to \mathcal{D}$ be covariant functors. A natural transformation α from F to G is a class of morphisms $\alpha_A\colon F(A) \to G(A)$, $A \in Ob(\mathcal{C})$ such that for every $A \xrightarrow{f} B$, $A, B \in Ob(\mathcal{C})$*

$$G(f) \circ \alpha_A = \alpha_B \circ F(f).$$

In other words, the following diagram is commutative:

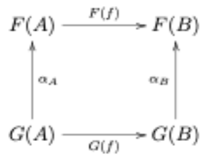


If α_A are all isomorphisms, we say α is a natural equivalence.

Definition 8.16. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, let $F, G: \mathcal{C} \to \mathcal{D}$ be contravariant functors. A natural transformation α from F to G is a class of morphisms $\alpha_A: F(A) \to G(A)$, $A \in Ob(\mathcal{C})$ such that for every $A \xrightarrow{f} B$, $A, B \in Ob(\mathcal{C})$*

$$F(f) \circ \alpha_A = \alpha_B \circ G(f).$$

In other words, the following diagram is commutative:



If α_A are all isomorphisms, we say α is a natural equivalence.

Example 8.17.

1. For any category $\mathcal{C}$ and any functor $F: \mathcal{C} \to \mathcal{C}$ the assignment $I(A) = I_A$, $A \in Ob(\mathcal{C})$, where $I_A: F(A) \to F(A)$ is given by

$$I_A = 1_{F(A)}$$

is a natural equivalence between F and F called the **identity natural equivalence.**

2. For the category $R - \mathcal{M}\text{od}$ consider the identity functor $I_{R-\mathcal{M}\text{od}}: R - \mathcal{M}\text{od} \to R - \mathcal{M}\text{od}$ given by

$$I_{R-\mathcal{M}od}(A) = A, A \in Ob\,(R - \mathcal{M}\text{od}) \text{ and } I_{R-\mathcal{M}od}(f) = f, A \xrightarrow{f} A'$$

and the double dual functor $^{**}: R - \mathcal{M}\text{od} \to R - \mathcal{M}\text{od}$ given by

$$A^{**} = (A^*)^* = \text{Hom}(\text{Hom}(A, R), R), A \in Ob\,(R - \mathcal{M}\text{od})$$

and

$$f^{**} = (f^*)^*, A \xrightarrow{f} A'$$

where $f^{**}: A^{**} \to A'^{**}$ is given by

$$f^{**}(\Psi) = \Psi \circ f^*, \text{Hom}(A, R) = A^* \xrightarrow{\Psi} R$$

and $f^*: A'^* \to A^*$ is given by

$$f^*(\psi) = \psi \circ f, A' \xrightarrow{\psi} R.$$

The assignment $\theta(A) = \theta_A$, $A \in R - \mathcal{M}\text{od}$, where $\theta_A: A \to A^{**}$ is given by

$$\theta_A(a) = a^*$$

where $a^*: \text{Hom}(A, R) = A^* \to R$ is given by

$$a^*(\psi) = \psi(a), A \xrightarrow{\psi} R$$

is a natural transformation from $I_{R-\mathcal{M}od}$ to **. If $R = F$ is a field and $R - \mathcal{M}\text{od}$ becomes the category of finitely dimensional F-vector spaces, then θ is a natural equivalence (which corresponds to the fact that the vector spaces V and V^{**} are isomorphic, where, for a given basis $v_1, ..., v_n$ of V, $v_1^*, ..., v_n^*$ becomes a basis of V^{**}).

Definition 8.18. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A covariant functor $F: \mathcal{C} \to \mathcal{D}$ is* ***faithful*** *if for all objects $A, B \in \text{Ob}(\mathcal{C})$ the induced function*

$$\text{Hom}_{\mathcal{C}}(A, B) \to \text{Hom}_{\mathcal{D}}(F(A), F(B))$$

is injective. If, moreover, it is surjective, then F shall be called ***fully faithful****.*

Proposition 8.19. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, let $F: \mathcal{C} \to \mathcal{D}$ be a fully faithful functor. Then, for all objects $A, B \in \text{Ob}(\mathcal{C})$, $A \cong B$ if and only if $F(A) \cong F(B)$.*

Proof. Fix two objects $A, B \in \text{Ob}(\mathcal{C})$ and assume that $A \cong B$. Let $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$ be two morphisms such that $f \circ g = 1_B$ and $g \circ f = 1_A$. Then

$$1_{F(B)} = F(1_B) = F(f \circ g) = F(f) \circ F(g) \text{ and } 1_{F(A)} = F(1_A) = F(g \circ f) = F(g) \circ F(f)$$

so that the morphisms $F(A) \xrightarrow{F(f)} F(B)$ and $F(B) \xrightarrow{F(g)} F(A)$ establish the isomorphism $F(A) \cong F(B)$.

Conversely, assume $F(A) \cong F(B)$ and let $F(A) \xrightarrow{\varphi} F(B)$ and $F(B) \xrightarrow{\psi} F(A)$ be two morphisms such that $\varphi \circ \psi = 1_{F(B)}$ and $\psi \circ \varphi = 1_{F(A)}$. Since the maps $\text{Hom}_{\mathcal{C}}(A, B) \to \text{Hom}_{\mathcal{D}}(F(A), F(B))$ and $\text{Hom}_{\mathcal{C}}(B, A) \to \text{Hom}_{\mathcal{D}}(F(B), F(A))$ are surjective, there exist morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$ such that $\varphi = F(f)$ and $\psi = F(g)$. Thus

$$1_{F(B)} = \varphi \circ \psi = F(f) \circ F(g) = F(f \circ g) \text{ and } 1_{F(A)} = \psi \circ \varphi = F(g) \circ F(f) = F(g \circ f).$$

On the other hand, $F(1_A) = 1_{F(A)}$ and $F(1_B) = 1_{F(B)}$. Since the maps $\text{Hom}_{\mathcal{C}}(A, B) \to \text{Hom}_{\mathcal{D}}(F(A), F(B))$ and $\text{Hom}_{\mathcal{C}}(B, A) \to \text{Hom}_{\mathcal{D}}(F(B), F(A))$ are injective, this yields

$$f \circ g = 1_B \text{ and } g \circ f = 1_A. \qquad \square$$

Definition 8.20. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A covariant functor $F: \mathcal{C} \to \mathcal{D}$ is called an* ***equivalence of categories*** *if it is fully faithful and essentially surjective, that is for every object $B \in \text{Ob}(\mathcal{D})$ there is an object $A \in \text{Ob}(\mathcal{C})$ such that $F(C) = D$.*