7 Categories. Products, coproducts, free objects. Initial, terminal and zero objects.

Definition 7.1. *A **category** $\mathcal{C}$ consists of a class of **objects** $Ob(\mathcal{C})$, denoted by $A, B, C, \ldots$ and a class of **morphisms** (or **arrows**) $Ar(\mathcal{C})$ together with:*

1. *a class of pairwise disjoint classes $Hom(A, B)$, one for each pair of objects $A, B \in Ob(\mathcal{C})$; an element f of the set $Hom(A, B)$ is called a **morphism** from A to B and denoted by $A \xrightarrow{f} B$ or $f: A \to B$,*

2. *functions $Hom(B, C) \times Hom(A, B) \to Hom(A, C)$, for each triple $A, B, C \in Ob(\mathcal{C})$, called **composition of morphisms**; for morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ the values of this function are denoted by $(g, f) \mapsto g \circ f$ and the morphism $A \xrightarrow{g \circ f} C$ is called the **composition** of morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$.*

Moreover, the following axioms are satisfied:

Associativity. *if $A \xrightarrow{f} B$, $B \xrightarrow{g} C$ and $C \xrightarrow{h} D$ are morphisms in $\mathcal{C}$, then*

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Identity. *for each object $B \in \mathcal{C}$ there exists a morphism $B \xrightarrow{1_B} B$ such that, for each $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$:*

$$1_B \circ f = f \text{ and } g \circ 1_B = g.$$

*If $Ob(\mathcal{C})$ and $Ar(\mathcal{C})$ are not proper classes, but sets, the category $\mathcal{C}$ is called **small**. If $Hom(A, B)$, for each pair of objects $A, B \in Ob(\mathcal{C})$, are not proper classes, but sets, the category $\mathcal{C}$ is called **locally small.***

*An **isomorphism** is a morphism $A \xrightarrow{f} B$ such that there is a morphism $B \xrightarrow{g} A$ such that*

$$f \circ g = 1_B \text{ and } g \circ f = 1_A.$$

*If there is an isomorphism $A \xrightarrow{f} B$ between the objects A and B, then the objects A and B are called **isomorphic** and denoted $A \cong B$.*

*An **automorfphism** is an isomorphism $A \xrightarrow{f} A$. An **endomorphism** is a morphism $A \xrightarrow{f} A$.*

Example 7.2.

1. The class $\mathcal{S}$et of all sets is a category, where morphisms are functions, and composition of morphisms is just composition of functions.

2. The class $\mathcal{G}$rp of all groups is a category, where morphisms are group homomorphisms, and composition of morphisms is just composition of functions.

3. The class $\mathcal{A}b$ of all Abelian groups is a category, where morphisms are group homomorphisms, and composition of morphisms is just composition of functions.

4. The class $\mathcal{R}$ng of all rings is a category, where morphisms are ring homomorphisms, and composition of morphisms is just composition of functions.

5. Let R be a ring. The class $R-\mathcal{M}$od of all left R-modules is a category, where morphisms are module homomorphisms, and composition of morphisms is just composition of functions.

6. The class $\mathcal{T}$op of all topological functions is a category, where morphisms are continuous maps, and composition of morphisms is just composition of functions.

7. The class $\mathcal{M}$etr of all metric spaces is a category, where morphisms are contactions, and composition of morphisms is just composition of functions; recall that a contraction of a metric space (X, ρ_X) to (Y, ρ_Y) is a function $f\colon X \to Y$ such that

$$\forall x, y \in X\, [\rho_Y(f(x), f(y)) \leq \rho_X(x, y)].$$

8. Let $(G, \cdot)$ be a group. Then G is a category, where the only object is the set G, and morphisms from the set Hom $G, G)$ are all elements of the group G, and composition of morphisms is just the group multiplication in G.

9. Let $(P, \leq)$ be a preorder, that is a nonempty set P together with the relation $\leq$ which is reflexive and transitive. Then P is a category, whose objects are elements of the set P, and the set of morphisms from $\mathrm{Hom}(a, b)$ contains at most one element which is defined by the condition:

$$a \to b \text{ if and only if } a \leq b.$$

   Since the relation $\leq$ is transitive, composition of morphisms is well-defined.

10. Let n be an ordinal number, that is the order type of a well-ordered set; recall that a partial ordering $(P, \leq)$ (i.e. a preorder where the relation $\leq$ is antisymmetric) is well-ordered if the relation $\leq$ is total and every non-empty subset S of P has the least element with respect to $\leq$, and two partial orders $(P_1, \leq_1)$ and $(P_2, \leq_2)$ are of the same type, if there is a bijection $f\colon P_1 \to P_2$ such that

$$\forall a, b \in P_1\, [a \leq_1 b \Rightarrow f(a) \leq_2 f(b)];$$

   having the same order type defines an equivalence relations, whose equivalence classes are called order types. As a finite ordinal number n we understand the well-ordered set of ordinal numbers preceeding n:

$$n = \{0, 1, 2, ..., n-1\},$$

   0 is understood to be the empty set, and the first infinite ordinal number ω (i.e. the order type of the set $\langle \mathrm{N} \rangle$) is understood as the well-ordered set:

$$\omega = \{0, 1, 2, ...\}.$$

   In particular, every ordinal number is a category. For example 3 is a category, whose objects are elements of the set $\{0, 1, 2\}$ and morphisms are defined as arrows

$$0 \to 1 \to 2$$

   together with their compositions. Likewise, ω is a category, whose objects are elements of the set $\{0, 1, 2, ...\}$, and morphisms are defined as arrows

$$0 \to 1 \to 2 \to 3 \to ...$$

   together with their compositions.

11. The class $\mathcal{O}$rd of all finite ordinal numbers forms a category, whose morphisms are functions preserving partial orderings, that is such that for two finite ordinal numbers $m = \{0, 1, 2, ..., m-1\}$ and $n = \{0, 1, 2, ..., n-1\}$, functions $f: m \to n$ such that

$$\forall i, j \in m\, [i \leq j \Rightarrow f(i) \leq f(j)].$$

Since the composition of two functions preserving an ordering is a function preserving an ordering, the composition of morphisms is well-defined. The category $\mathcal{O}$rd, depending on context, is also denoted by Δ and called the simplex category.

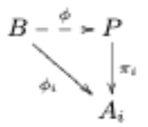
12. Let $\mathcal{C}$ be any category. The class $\mathcal{C}^{\to}$ of all arrows of the category $\mathcal{C}$ is a category, whose objects are morphisms of the category $\mathcal{C}$, and morphisms are defined as follows: if $f, g \in Ob(\mathcal{C}^{\to})$, where $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$, then $\mathrm{Hom}(f, g)$ consists of all pairs (ϕ, ψ) such that $A \xrightarrow{\phi} C$, $B \xrightarrow{\psi} D$ and $\psi \circ f = g \circ \phi$.

13. Let $\mathcal{I}$ be any class. The class $\mathcal{I}$ is a category whose objects are elements of $\mathcal{I}$, and morphisms are the identity maps. This category is called the discrete category of $\mathcal{I}$.

Definition 7.3. *Let $\mathcal{C}$ be a category, let $\{A_i : i \in I\}$ be a family of objects of $\mathcal{C}$. A* ***product*** *of the family $\{A_i : i \in I\}$ is an object P together with a family of morphisms $\{P \xrightarrow{\pi_i} A_i : i \in I\}$ such that for every object B and every family of morphisms $\{B \xrightarrow{\phi_i} A_i : i \in I\}$ there exists exactly one morphism $B \xrightarrow{\phi} P$ such that*

$$\pi_i \circ \phi = \phi_i,$$

for $i \in I$. In other words, the following diagram is commutative:



The product P is denoted by $\prod_{i \in I} A_i$.

Example 7.4.

1. Prodcucts exist in the category $\mathcal{S}$et, these are cartesian products with projections.

2. Prodcucts exist in the category $\mathcal{G}$rp, these are products of groups with canonical epimorphisms.

3. Prodcucts exist in the category $\mathcal{A}b$, these are products of Abelian groups with canonical epimorphisms.

4. Prodcucts exist in the category $\mathcal{R}$ng.

5. Prodcucts exist in the category $R - \mathcal{M}$od.

6. Prodcucts exist in the category $\mathcal{T}$op.

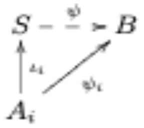
7. Prodcucts don't exist in the category $\mathcal{M}$etr.

8. Prodcucts exist in the category $\mathcal{O}$rd.

Definition 7.5. *Let $\mathcal{C}$ be a category, let $\{A_i : i \in I\}$ be a family of objects of $\mathcal{C}$. A* ***coproduct*** *of the family $\{A_i : i \in I\}$ is an object S together with a family of morphisms $\{A_i \xrightarrow{\iota_i} S : i \in I\}$ such that for every object B and every family of morphisms $\{A_i \xrightarrow{\psi_i} S : i \in I\}$ there exists exactly one morphism $S \xrightarrow{\psi} B$ such that*

$$\psi \circ \iota_i = \psi_i,$$

for $i \in I$. In other words, the following diagram is commutative:



The coproduct S is denoted by $\coprod_{i \in I} A_i$.

Example 7.6.

1. Coprodcucts exist in the category $\mathcal{S}$et, these are disjoint unions of sets together with inclusions.
2. Coprodcucts exist in the category $\mathcal{A}b$, these are direct sums of Abelian groups with canonical monomorphisms.
3. Coprodcucts exist in the category $\mathcal{G}$rp, these are free products of groups with canonical monomorphisms.
4. Coprodcucts exist in the category $R-\mathcal{M}$od.

1. Coprodcucts exist in the category $\mathcal{S}$et, these are disjoint unions of sets together with inclusions.
2. Coprodcucts exist in the category $\mathcal{A}b$, these are direct sums of Abelian groups with canonical monomorphisms.
3. Coprodcucts exist in the category $\mathcal{G}$rp, these are free products of groups with canonical monomorphisms.
4. Coprodcucts exist in the category $R-\mathcal{M}$od.

Definition 7.7. *Let $\mathcal{C}$ be a category. The category $\mathcal{C}$ is called* **concrete,** *if there is a function σ: $Ob(\mathcal{C}) \rightarrow Ob(\mathcal{S}\mathrm{et})$ such that*

1. *every morphism $A \xrightarrow{f} B$ is a function between sets $f : \sigma(A) \rightarrow \sigma(B)$;*
2. *the identity morphism $A \xrightarrow{1_A} A$ is the identity function $1_A : \sigma(A) \rightarrow \sigma(A)$;*
3. *composition of morphisms is just the composition of functions.*

Example 7.8.

1. The categories $\mathcal{S}et$, $\mathcal{A}b$, $\mathcal{G}rp$, $R-\mathcal{M}od$ are concrete.
2. A group G viewed as a category with one object is not concrete.

Definition 7.9. *Let $\mathcal{C}$ be a concrete category, let F be an object of the categoryi $\mathcal{C}$, let X be a nonempty set, let $f: X \to F$ be a map between sets. The object F is called **free with basis X** if and only if for every object H and every map between sets $h: X \to H$ there exists exactly one morphism $F \xrightarrow{\phi} H$ such that $\phi \circ f = h$.*

Example 7.10.

1. Free objects exist in the category $\mathcal{G}rp$, these are free groups.

2. Free objects exist in the category $\mathcal{A}b$, these are free Abelian groups.

3. Free objects exist in the category of unitary modules over a fixed ring with 1, these are free modules.

Definition 7.11. *Let $\mathcal{C}$ be a category, let $B,\ C \in Ob(\mathcal{C})$. A morphism $B \xrightarrow{\phi} C$ is called a **monomorphism** if for any object A and morphisms $A \underset{\psi_2}{\overset{\psi_1}{\rightrightarrows}} B$:*

$$\text{if } \phi \circ \psi_1 = \phi \circ \psi_2 \text{ then } \psi_1 = \psi_2.$$

*A morphism $B \xrightarrow{\phi} C$ is called an **epimorphism** if for any object D and morphisms $C \underset{\psi_2}{\overset{\psi_1}{\rightrightarrows}} D$:*

$$\text{if } \psi_1 \circ \phi = \psi_2 \circ \phi \text{ then } \psi_1 = \psi_2.$$

Example 7.12.

1. Consider the category $\mathcal{G}rp$. Then a morphism $B \xrightarrow{\phi} C$ is a monomorphism if and only if it is an injective homomorphism of groups, and is an epimorphism, if it is a surjective homomorphism of groups.

2. Consider the category $\mathcal{R}ng$. Then a morphism $B \xrightarrow{\phi} C$ is a monomorphism if and only if it is an injective homomorphism of rings.

3. Consider the category $R-\mathcal{M}od$. Then a morphism $B \xrightarrow{\phi} C$ is a monomorphism if and only if it is an injective homomorphism of modules, and is an epimorphism, if it is a surjective homomorphism of modules.

Remark 7.13. Let $\mathcal{C}$ be a category, let $B, C, D \in Ob(\mathcal{C})$, let $B \xrightarrow{\phi} C$ and $C \xrightarrow{\psi} D$ be morphisms. Then:

1. if ϕ i ψ are monomorphisms, then $\psi \circ \phi$ is a monomorphism

2. if $\psi \circ \phi$ is a monomorphism, then ϕ is a monomorphism

3. if ϕ i ψ are epimorphisms, then $\psi \circ \phi$ is an epimorphism

4. if $\psi \circ \phi$ is an epimorphism, then ψ is an epimorphism

5. if ϕ is an isomorphism, then it is a monomorphism and an epiorphism.

The proof is left to the reader as an excercise.

Definition 7.14. *Let $\mathcal{C}$ be a category. An object I of $\mathcal{C}$ is called an* ***initial object*** *(or* ***universal****), if, for each object C of $\mathcal{C}$ there is exactly one morphism $I \xrightarrow{i} C$.*

An object T of $\mathcal{C}$ is called a ***terminal object*** *(or* ***couniversal****), if, for each object C of $\mathcal{C}$ there is exactly one morphism $C \xrightarrow{t} T$.*

An object Z of $\mathcal{C}$ is called a ***zero object*** *if it is both initial and terminal.*

Example 7.15.

1. Consider the category $\mathcal{G}$rp. The trivial group $\{1\}$ is both an initial and a terminal object.

2. Consider the category $\mathcal{A}b$. The trivial group $\{1\}$ is both an initial and a terminal object

3. Consider the category $R - \mathcal{M}$od. The trivial module $\{0\}$ is both an initial and a terminal object

Remark 7.16. Let $\mathcal{C}$ be a category, let C be an object of $\mathcal{C}$.

1. Any two initial (terminal, zero) objects are isomorphic.

2. If 0 is a zero object, then the unique morphism $0 \to C$ is a monomorphism, and the unique morphism $C \to 0$ is an epimorphism.

The proof is left to the reader as an excercise.