9 Rational maps of affine algebraic sets. Birational equivalence of affine algebraic sets.

Definition 9.1. *Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes. A* ***rational map*** $f: V \to W$ *is a map such that there exist* $f_1, \ldots, f_m \in k(V)$ *such that* $f(a) = (f_1(a), \ldots, f_m(a))$*, for all the points* $a \in V$ *where all the rational functions* $f_1, \ldots, f_m \in k(V)$ *are defined.*

Remark 9.2. Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes, let $f\colon V \to W$ be a rational map, $f = (f_1, \ldots, f_m)$ with $f_1, \ldots, f_m \in k(V)$. There exists an open set $\emptyset \neq U \subseteq V$ such that $f_1 \restriction_U, \ldots, f_m \restriction_U$ are regular on U. In other words, we can think of rational maps as defined on open subsets.

Remark 9.3. Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes, let $f_1, \ldots, f_m \in k(V)$. Then $f_1, \ldots,$ f_m define a rational map $f: V \to W$.

Remark 9.4. Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes, let $f: V \to W$ be a rational map and assume that $f(V)$ is dense in W. The map f defines a field embedding $f^*: k(W) \to k(V)$.

Definition 9.5. *Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes, let $f\colon V \to W$ be a rational map such that $f(V)$ is dense in W. The map f is a **birational equivalence** if there is a rational map $g\colon W \to V$ such that $g(W)$ is dense in V and*

$$f \circ g = 1_W \ \textit{and} \ g \circ f = 1_V.$$

*In this case we say that V and W are **birationally equivalent** or **birational**.*

Corollary 9.6. *Let* $V \subseteq k^n$ *and* $W \subseteq k^m$ *be affine varietes. Then* V *and* W *are birationally equivalent if and only if* $k(V) \cong k(W)$.

Example 9.7. Let $V = \mathcal{Z}(xy-1)$ and $W = \mathcal{Z}(y)$, let $f\colon V \to W$ be given by $(x, y) \mapsto (x, 0)$. This is a birational equivalence, but not an isomorphism.

Example 9.8. Let $V = \mathcal{Z}(y)$ and $W = \mathcal{Z}(y^2 - x^3)$, let $f\colon V \to W$ be given by $(x,0) \mapsto (x^2, x^3)$. This is a birational equivalence (the inverse map $g\colon W \to V$ being $(x, y) \mapsto \left(\frac{y}{x}, 0\right)$), but not an isomorphism.

Proposition 9.9. *Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes, let $f\colon V \to W$ be a birational equivalence. Then there exist open subsets $U \subseteq V$ and $U' \subseteq W$ which are isomorphic.*

Proposition 9.10. (Noether normalization lemma) *Let k be algebraically closed, and $k \subseteq K$ a finitely generated field extension. Then there exist elements $z_1, \ldots, z_{d+1} \in K$ with $K = k(z_1, \ldots, z_{d+1})$ such that $z_1, \ldots, z_d$ are algebraically independent over k, and z_{d+1} is separable over $k(z_1, \ldots, z_{d+1})$.*

Proposition 9.11. *Let $V \subseteq k^n$ be an affine variety. Then V is birationally equivalent to a hypersurface of some affine space k^m.*

A variety is called **rational** if it is birationally equivalent to k^n, for some n.