8 Rational functions field of an affine algebraic variety.

Definition 8.1. *Let $V \subseteq k^n$ be an affine algebraic variety. The field of fractions of the coordinate ring $k[V]$ will be called the* ***field of rational functions*** *of V and denoted by $k(V)$, and its elements* ***rational functions*** *on V.*

Example 8.2.

- $V = \{(a_1, ..., a_n)\} \in k^n$, $k(V) \cong k$;
- $V = k^k$, $k(V) \cong k(x_1, ..., x_n)$.

Definition 8.3. *Let $V \subseteq k^n$ be an affine algebraic variety. A rational function $\varphi \in k(V)$ is* ***defined*** *at a point $(a_1, \ldots, a_n) \in V$ if $\varphi = \frac{f}{g}$, for some $f, g \in k[V]$ with $g(a_1, \ldots, a_n) \neq 0$. In this case we say that $\frac{f(a_1, \ldots, a_n)}{g(a_1, \ldots, a_n)} \in k$ is the* ***value*** *of φ at $(a_1, \ldots, a_n)$, and denote it by $\varphi(a_1, \ldots, a_n)$.*

Remark 8.4. Let $V \subseteq k^n$ be an affine algebraic variety, let $\varphi \in k(V)$ be defined at $(a_1, \ldots, a_n) \in V$. The value of φ at $(a_1, \ldots, a_n)$ is uniquely defined.

Example 8.5. Let $V = \mathcal{Z}(x^2 + y^2 - 1) \subseteq \mathbb{C}^2$. Then $\mathbb{C}(V) \cong \mathbb{C}(x, y)$ with $x^2 + y^2 = 1$. Let $\varphi = \frac{1-y}{x} \in \mathbb{C}(V)$. Then φ is defined at $(0, 1) \in V$ and $\varphi(0, 1) = 0$, but φ is not defined at $(0, -1)$.

Remark 8.6. Let $V \subseteq k^n$ be an affine algebraic variety. Every element $\frac{f}{g} \in k(V)$, $f, g \in k[V]$, determines a rational function defined on some nonempty open subset $U \subseteq V$ with values in k.

Remark 8.7. Let $V \subseteq k^n$ be an affine algebraic variety. If the rational functions $\varphi_1, \varphi_2 \in k(V)$ have the same values on a certain nonempty open subset of $U \subseteq V$, then they are equal.

Theorem 8.8. *Let* $V \subseteq k^n$ *be an affine algebraic variety. If the rational function* $\frac{f}{g} \in k(V)$, f, $g \in k[V]$, *is defined at every point of* V, *then* $\frac{f}{g} = \frac{h}{1}$, *for some* $h \in k[V]$.

Definition 8.9. *Let $V \subseteq k^n$ be an affine algebraic set, let $V = V_1 \cup \ldots \cup V_m$ be the decomposition of V into affine algebraic varieties. The* ***k-algebra of rational functions*** *of V is defined to be*

$$k(V) = k(V_1) \oplus \ldots \oplus k(V_m)$$

and its elements are called ***rational functions*** *on V.*

Definition 8.10. *Let* $V \subseteq k^n$ *be an affine algebraic set. If a rational function* $\varphi \in k(V)$ *is defined at every point of an open subset* $U \subseteq V$*, then the restriction* $\varphi \restriction_U$ *wil be called a* ***regular*** *function on* U*.*

Example 8.11. Let $V = \mathcal{Z}(xy)$. Then $V = \mathcal{Z}(x) \cup \mathcal{Z}(y)$. Let $f = x(y+1)$. Then $f\restriction_{\mathcal{Z}(x)\setminus\{(0,0)\}}=0$ and $f\restriction_{\mathcal{Z}(y)\setminus\{(0,0)\}}=1$, $f \in k(V)$, f is regular on both $\mathcal{Z}(x)$ and $\mathcal{Z}(y)$, but not regular on V, as it is not defined on $(0,0)$.

Remark 8.12. Let $V \subseteq k^n$ be an affine algebraic set, let $f \in k(V)$. Then f is continuous on the set of points where it is defined.

Theorem 8.13. *Let $V \subseteq k^n$ be an affine algebraic variety, let $f \in k[V] \setminus \{0\}$, let*

$$k[V]_f = \left\{ h \in k(V) \,|\, h = \frac{g}{f}, m \in \mathbb{Z}, g \in k[V] \right\}$$

and

$$V_f = \{(a_1, ..., a_n) \in V \,|\, f(a_1, ..., a_n) \neq 0\}.$$

Then the k-algebra of regular functions on V_f is isomorphic to $k[V]_f$.