7 Coordinate ring of an affine algebraic set. Morphisms of affine algebraic sets. Category of affine algebraic sets.

7.1 Coordinate ring of an affine algebraic set.

Definition 7.1. *Let k be a field, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V. The ring $k[V] := k[x_1, \ldots, x_n]/\mathcal{I}(V)$ is called the **coordinate ring** of V.*

Remark 7.2. Let k be a field, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V. Let $f \in k[x_1, \ldots, x_n]$. The polynomial f defines a polynomial function $k^n \to k$. Let f_V be the restriction of f to the set V, $f_V = f \restriction_V$. Then $f_V = g_V$ if and only if $f + \mathcal{I}(V) = g + \mathcal{I}(V)$.

Remark 7.3. Let k be a field, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V. Let κ: $k[x_1, \ldots, x_n] \to k[V]$ be the canonical epimorphism, $\kappa(f) = \overline{f} := f + \mathcal{I}(V)$. Then $k[V]$ is a k-ring finitely generated over k by $\overline{x_1}, \ldots, \overline{x_2}$.

Remark 7.4. Let k be algebraically closed, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V. Then $k[V]$ has no nonzero nilpotents.

Theorem 7.5. *Let k be algebraically closed. Then a k-ring A is isomorphic to a coordinate ring of an affine algebraic set $V \subseteq k^n$ if and only if it is finitely generated over k and has no nonzero nilpotents.*

Example 7.6.

- $V = k^n$, $k[V] \cong k[x_1, ..., x_n]$;
- $V = \emptyset$, $k[V] \cong 0$;
- $V = \{(a_1, ..., a_n)\}$, $k[V] \cong k$.

Example 7.7. $V = \mathcal{Z}(f)$, $f \in k[x_1, \ldots, x_n]$ is square-free. $k[V] \cong k[x_1, \ldots, x_n] / (f) \cong k[\alpha_1, \ldots, \alpha_n]$ where $f(\alpha_1, \ldots, \alpha_n) = 0$.

Example 7.8. $V = \mathcal{Z}(a_1x_1 + \ldots + a_nx_n - b)$, $k[V] \cong k[x_1, \ldots, x_{n-1}]$.

7.2 Basic notions in category theory.

Definition 7.9. *A **category** $\mathcal{C}$ consists of a class of **objects** $\mathrm{Ob}(\mathcal{C})$, denoted by $A, B, C, \ldots$ and a class of **morphisms** (or **arrows**) $\mathrm{Ar}(\mathcal{C})$ together with:*

1. *classes of pairwise disjoint arrows $\mathrm{Hom}(A, B)$, one for each pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$; and elements f of the class $\mathrm{Hom}(A, B)$ shall be called a **morphism** from A to B and denoted by $A \xrightarrow{f} B$ or $f: A \to B$,*
2. *functions $\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$, for each triple of objects A, B, $C \in \mathrm{Ob}(\mathcal{C})$, called **composition** of morphisms; for morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ values of this function shall be denoted by $(g, f) \mapsto g \circ f$, and the morphism $A \xrightarrow{g \circ f} C$ shall be called the composition of morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$.*

Moreover, we require that the following two axioms hold true:

Associativity. *If* $A \xrightarrow{f} B$, $B \xrightarrow{g} C$ *and* $C \xrightarrow{h} D$ *are morphisms in* $\mathcal{C}$, *then*

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Identity. *For every object* $B \in \mathrm{Ob}(\mathcal{C})$ *there exists a morphism* $B \xrightarrow{1_B} B$ *such that for all morphisms* $A \xrightarrow{f} B$ *and* $B \xrightarrow{g} C$

$$1_B \circ f = f \;\; and \; g \circ 1_B = g.$$

If the classes $\mathrm{Ob}(\mathcal{C})$ *and* $\mathrm{Ar}(\mathcal{C})$ *are sets, we shall call the category* $\mathcal{C}$ ***small****. If all classes* $\mathrm{Hom}(A, B)$ *are sets, we shall call the category* $\mathcal{C}$ ***locally small.***

Example 7.10.

1. We shall set the notation for a number of familiar categories here:
   - $\mathcal{S}\mathrm{et}$ is the category of sets with functions as morphisms;
   - $\mathcal{G}\mathrm{rp}$ is the category of groups with group homomorphisms as morphisms;
   - $\mathcal{T}\mathrm{op}$ is the category of topological spaces with continuous functions as morphisms.
   - $\mathcal{A}b$ is the category of Abelian groups;
   - $\mathcal{R}\mathrm{ng}$ is the category of rigns;
   - $\mathcal{F}\mathrm{ield}$ is the category of fields;
   - $k - \mathcal{V}\mathrm{ect}$ is the category of $k-$vector spaces;
   - $k - \mathcal{A}\mathrm{lg}^{0}_{\mathrm{fg}}$ is the category of finitely generated k-algebras over an algebraically closed field k with no nonzero nilpotent elements.

   All these categories are locally small, but not small.

2. The notion of a category allows for a different take on familiar constructions in mathematics. For example, consider a partial order $(P, \leqslant)$. One checks that considering the elements of P as objects, and defining morphisms by

$$a \to b \qquad \Leftrightarrow \qquad a \leqslant b$$

one obains a category, which is small provided P is a set.

3. **Posetal category of a topological space.** A special case of the above construction that we shall frequently use is the following one. Let $(X\,,\tau)$ be a topological space. In particular, $(\tau\,,\subseteq)$ is a partial order which, viewed as a category, shall be denoted by $\mathcal{O}(X)$ and called the posetal category of the space $(X\,,\tau)$.

Definition 7.11. *Let $\mathcal{C}$ be a category. If, for two objects $A, B \in \mathrm{Ob}(\mathcal{C})$ there exist morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$ such that*

$$f \circ g = 1_B \ \ and \ \ g \circ f = 1_A$$

*then we say that objects A and B are **isomorphic** and write $A \cong B$.*

Definition 7.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **covariant functor** F from $\mathcal{C}$ to $\mathcal{D}$ is a pair of maps $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and $\mathrm{Ar}(\mathcal{C}) \to \mathrm{Ar}(\mathcal{D})$ (denoted by the same symbol F), that assign to each object $A \in \mathrm{Ob}(\mathcal{C})$ an object $F(A) \in \mathrm{Ob}(\mathcal{D})$ and to each morphism $A \xrightarrow{f} B$ in $\mathrm{Ar}(\mathcal{C})$ a morphism $F(A) \xrightarrow{F(f)} F(B)$ in $\mathrm{Ar}(\mathcal{D})$ in a way that the following two axioms are satisfied:*

1. $F(1_A) = 1_{F(A)}$, *for every object $A \in \mathrm{Ob}(\mathcal{C})$;*
2. $F(g \circ f) = F(g) \circ F(f)$, *for all arrows $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ in $\mathrm{Ar}(\mathcal{C})$.*

A ***contravariant functor*** *is defined in an analogous way, but to each morphism* $A \xrightarrow{f} B$ *in* $\mathrm{Ar}(\mathcal{C})$ *a morphism* $F(B) \xrightarrow{F(f)} F(A)$ *in* $\mathrm{Ar}(\mathcal{D})$ *is assigned and the axiom 2. is replaced with:*

2'. $F(g \circ f) = F(f) \circ F(g)$, *for all arrow* $A \xrightarrow{f} B$ *and* $B \xrightarrow{g} C$ *in* $\mathrm{Ar}(\mathcal{C})$.

Example 7.13.

1. **Identity functors.** For every category $\mathcal{C}$ the map $I_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$ given by

$$I_{\mathcal{C}}(A) = A, \text{ for every object } A \in \mathrm{Ob}(\mathcal{C}), \qquad I_{\mathcal{C}}(f) = f, \text{ for every morphism } A \xrightarrow{f} B \text{ in } \mathcal{C}$$

   is a covariant functor that shall be called the identity functor.

2. **Forgetful functors.** The map $F\colon \mathcal{G}\mathrm{rp} \to \mathcal{S}\mathrm{et}$ given by

$$F(G) = G, \text{ for every object } G \in \mathrm{Ob}(\mathcal{G}\mathrm{rp}), \qquad F(f) = f, \text{ for every morphism } G \xrightarrow{f} H \text{ in } \mathcal{G}\mathrm{rp}$$

is a covariant functor that shall be called the forgetful functor. In the same way we can define forgetful functors $\mathcal{R}\mathrm{ng} \to \mathcal{S}\mathrm{et}$, $\mathcal{T}\mathrm{op} \to \mathcal{S}\mathrm{et}$ etc.

3. **Free functors.** The map $F\colon \mathcal{S}\mathrm{et} \to \mathcal{A}b$ given by

$$F(X) = \text{free Abelian group with basis } X, \text{ for every object } X \in \mathrm{Ob}(\mathcal{A}b),$$

and

$F(f) =$ the uniquely defined morphism $\overline{f}$ s.t. $\overline{f}\restriction_X = f$, for every morphism $X \xrightarrow{f} Y$ in $\mathcal{S}\mathrm{et}$

is a covariant functor that creates free Abelian groups. In the same way we can define free fuctors $\mathcal{S}\mathrm{et} \to \mathcal{G}\mathrm{rp}$ etc.

4. For a category $\mathcal{C}$ we define the **opposite category** $\mathcal{C}^{\mathrm{op}}$ as follows: $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$, and for $A, B \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(B, A)$$

and

$$f^{\mathrm{op}} \circ g^{\mathrm{op}} = (g \circ f)^{\mathrm{op}}.$$

For example if $\mathcal{C}$ consists of the following objects and morphisms:

$$A \to B \to C \to D,$$

then $\mathcal{C}^{\mathrm{op}}$ is of the following form:

$$A \leftarrow B \leftarrow C \leftarrow D.$$

If $F: \mathcal{C} \to \mathcal{D}$ is a contravariant functor, then $\bar{F}: \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ defined by

$$\bar{F}(A) = F(A), \text{ for every object } A \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}}), \bar{F}(f^{\mathrm{op}}) = F(f), \text{ for every morphism } A \xrightarrow{f} B \text{ in } \mathcal{C}$$

is a covariant functor.

Definition 7.14. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A covariant functor $F\colon\mathcal{C}\to\mathcal{D}$ is* ***faithful*** *if for all objects $A, B\in \mathrm{Ob}(\mathcal{C})$ the induced function*

$$\mathrm{Hom}_{\mathcal{C}}(A,B)\to \mathrm{Hom}_{\mathcal{D}}(F(A),F(B))$$

is injective. If, moreover, it is surjective, then F shall be called ***fully faithful****.*

Proposition 7.15. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, let $F\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor. Then, for all objects $A, B \in \mathrm{Ob}(\mathcal{C})$, $A \cong B$ if and only if $F(A) \cong F(B)$.*

Definition 7.16. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be categories. A covariant functor* $F\colon \mathcal{C} \to \mathcal{D}$ *is called a* ***equivalence of categories*** *if it is fully faithful and essentially surjective, that is for every object* $B \in \mathrm{Ob}(\mathcal{D})$ *there is an object* $A \in \mathrm{Ob}(\mathcal{C})$ *such that* $F(C) = D$*.*

7.3 Category of affine algebraic sets.

Definition 7.17. *Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine algebraic sets. A* ***morphism*** $f: V \to W$ *is a map such that there exist* $f_1, \ldots, f_m \in k[V]$ *such that* $f(a) = (f_1(a), \ldots, f_m(a))$*, for all* $a \in V$*.*

Remark 7.18. Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine algebraic sets, let $f_1, \ldots, f_m \in k[V]$. Then $f = (f_1, \ldots, f_m) \colon V \to W$ is a morphism if and only if

$$g(f_1, \ldots, f_m) = 0 \in k[V] \quad \text{for all } g \in \mathcal{I}(W).$$

Example 7.19.

- Let $f \in k[V]$. Then $f\colon V \to k$ is a morphism.
- Let $f\colon k^n \to k^m$ be a linear map. Then f is a morphism.
- Let $f\colon \mathcal{Z}(xy-1) \to k$ be given by $f(x,y)=x$. Then f is a morphism.
- Let $f\colon k \to \mathcal{Z}(y^2-x^3)$ be given by $f(t)=(t^2,t^3)$. Then f is a morphism.

Example 7.20. One easily checks that:

- $\mathcal{Z}(y - x^k) \cong k$ via $f(x, y) = x$ and $g(t) = (t, t^k)$;
- $f\colon \mathcal{Z}(xy - 1) \to k$ given by $f(x, y) = x$ is not an isomorphism;
- $f\colon k \to \mathcal{Z}(y^2 - x^3)$ given by $f(t) = (t^2, t^3)$ is not an isomorphism, even though it is a bijection.

- We shall write $k - \mathcal{A}\text{ff}$ for the category of affine algebraic sets over an algebraically closed field k with morphisms defined above.

Theorem 7.21. *Let k be algebraically closed and consider the categories $k-\mathcal{A}\mathrm{ff}$ and $k-\mathcal{A}\mathrm{lg}^0_{\mathrm{fg}}$. The assignment*

$$F(V)=k[V] \text{ for an affine algebraic set } V\subseteq k^n$$

and

$$F(\varphi)=\varphi^* \text{ for a morphism of affine algebraic sets } \varphi: V\to W,$$

where $\varphi^*: k[W]\to k[V]$ is given by the formula

$$\varphi^*(f)=g\circ\varphi$$

defines an equivalence of categories $k-\mathcal{A}\mathrm{ff}^{\mathrm{op}}$ and $k-\mathcal{A}\mathrm{lg}^0_{\mathrm{fg}}$.