3 Minimal primary decomposition.

3.1 Radical of an ideal.

Definition 3.1. *Let R be a ring, let $\mathfrak{a} \lhd R$. The* ***radical*** *of the ideal $\mathfrak{a}$ is defined to be*

$$\operatorname{rad}\mathfrak{a} = \{r \in R \mid \exists n \in \mathbb{N}\, r^n \in \mathfrak{a}\}.$$

Remark 3.2. Let R be a ring, let $\mathfrak{a} \lhd R$. Then $\operatorname{rad} \mathfrak{a}$ is an ideal.

Remark 3.3. Let R be a ring, let $\mathfrak{a}, \mathfrak{b} \lhd R$.

1. $\mathfrak{a} \subseteq \operatorname{rad} \mathfrak{a}$,
2. $\mathfrak{a} \subseteq \mathfrak{b} \Rightarrow \operatorname{rad} \mathfrak{a} \subseteq \operatorname{rad} \mathfrak{b}$,
3. $\operatorname{rad} (\operatorname{rad} \mathfrak{a}) = \operatorname{rad} \mathfrak{a}$,
4. $\operatorname{rad} \mathfrak{a} \cdot \mathfrak{b} = \operatorname{rad} \mathfrak{a} \cap \mathfrak{b}$,
5. $\operatorname{rad} \mathfrak{a} \cap \mathfrak{b} = \operatorname{rad} \mathfrak{a} \cap \operatorname{rad} \mathfrak{b}$,
6. $\operatorname{rad} \mathfrak{a} = (1) \Leftrightarrow \mathfrak{a} = (1)$,
7. $\operatorname{rad} \mathfrak{a} + \mathfrak{b} = \operatorname{rad}(\operatorname{rad} \mathfrak{a} + \operatorname{rad} \mathfrak{b})$,
8. $\mathfrak{a} + \mathfrak{b} = (1) \Leftrightarrow \operatorname{rad} \mathfrak{a} + \operatorname{rad} \mathfrak{b} = (1)$.

Remark 3.4. Let R be a ring, let $\mathfrak{p} \lhd R$ be a prime ideal, let $m \in \mathbb{N}$. Then $\operatorname{rad} \mathfrak{p}^m = \mathfrak{p}$.

Remark 3.5. Let R be a ring, let $\mathfrak{a} \lhd R$. If $\operatorname{rad} \mathfrak{a}$ is a maximal ideal, then $\mathfrak{a}$ is primary.

Lemma 3.6. *Let R be a ring, let $\mathfrak{q} \lhd R$ be a primary ideal. Then $\operatorname{rad} \mathfrak{q}$ is prime.*

Definition 3.7. *Let R be a ring, let $\mathfrak{q} \lhd R$ be a primary ideal and let $\mathfrak{p} = \operatorname{rad} \mathfrak{q}$. Then $\mathfrak{q}$ is called $\mathfrak{p}$-**primary**.*

Remark 3.8. Let R be a ring, let $\mathfrak{m} \lhd R$ be a maximal ideal, let $m \in \mathbb{N}$. Then $\mathfrak{m}^m$ is $\mathfrak{m}$-primary.

Lemma 3.9. *Let* R *be a ring, let* $\mathfrak{q}_1, ..., \mathfrak{q}_n$ *be* $\mathfrak{p}$*-primary. Then* $\mathfrak{q}_1 \cap ... \cap \mathfrak{q}_n$ *is* $\mathfrak{p}$*-primary.*

Definition 3.10. *Let R be a ring, let $\mathfrak{a} \lhd R$ be a proper ideal and let*

$$\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$$

be a primary decomposition of $\mathfrak{a}$. If

$$\mathfrak{q}_j \not\supseteq \bigcap_{i \neq j} \mathfrak{q}_i$$

and

$$\operatorname{rad} \mathfrak{q}_i \neq \operatorname{rad} \mathfrak{q}_j \ \text{ for } i \neq j,$$

then the primary decomposition $\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$ is called ***minimal****.*

Theorem 3.11. (Noether-Lasker) *Let R be a Noetherian ring, let $\mathfrak{a} \lhd R$ be a proper ideal. Then $\mathfrak{q}$ has a minimal primary decomposition and the prime ideals $\mathfrak{p}_i = \mathrm{rad}\ \mathfrak{q}_i$ are uniquely determined up to the order.*