10 Projective space. Projective algebraic sets.

10.1 Projective space.

The concept of a projective space originated from the visual effect of perspective, where parallel lines seem to meet at infinity. A projective space may thus be viewed as the extension of an Euclidean space in such a way that there is one point at infinity for each direction of parallel lines.

Consider the affine plane k^2. Each point $(x, y) \in k^2$ can be idetified with the point $(x, y, 1) \in k^3$. Every point $(x, y, 1) \in k^3$ determines a line in k^3 that passes through $(0, 0, 0)$ and $(x, y, 1)$. Every line through $(0, 0, 0)$ except those lying on the plane $z = 0$ corresponds to exactly one such point. The lines through $(0, 0, 0)$ in the plane $z = 0$ can be thought of as corresponding to the "points at infinity".

Definition 10.1. *Let k be a field.* ***Projective n-space over k****, written $\mathbb{P}^n(k)$, is defined to be the set of all lines through $(0, ..., 0) \in k^{n+1}$. Any point $(x_1, ..., x_{n+1}) \neq (0, ..., 0)$ determines a unique such line, namely $\{(\lambda x_1, ..., \lambda x_{n+1}) | \lambda \in k\}$. Elements of $\mathbb{P}^n(k)$ are called* ***points****. If a point $P \in \mathbb{P}^n(k)$ is determined as above by some $(x_1, ..., x_{n+1}) \in k^{n+1} \setminus \{(0, ...0)\}$, we say that $(x_1, ..., x_{n+1})$ are the* ***homogeneous coordinates*** *of P and write $P = [x_1 \colon ... \colon x_{n+1}]$.*

Remark 10.2. Let $\mathbb{P}^n(k)$ be a projective n-space over k. Two points $(x_1, ..., x_{n+1})$, $(y_1, ..., y_{n+1}) \in k^{n+1}$ determine the same line is and only if there is a nonzero $\lambda \in k$ such that $x_1 = \lambda y_1, ..., x_{n+1} = \lambda y_{n+1}$. Let us say that $(x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}) \in k^{n+1}$ are equivalent if this is the case. Then $\mathbb{P}^n(k)$ may be identified with the set of equivalence classes of points $k^{n+1} \setminus \{(0, ...0)\}$.

Remark 10.3. Let $\mathbb{P}^n(k)$ be a projective n-space over k and let $P \in \mathbb{P}^n(k)$, $P = [x_1 \colon ... \colon x_{n+1}]$. Note that the i-th coordinate x_i is not well-defined, but that it is a well-defined notion to say whether the i-th coordinate is zero or nonzero. If $x_i \neq 0$ the ratios x_j / x_i are well-defined, since they are unchanged under the abovedescribed equivalence.

Definition 10.4. *Let $\mathbb{P}^n(k)$ be a projective n-space over k. Let*

$$U_i = \{[y_1 \colon ... \colon y_{n+1}] \in \mathbb{P}^n(k) | \, y_i \neq 0\}$$

Each $P \in U_i$ can be written uniquely in the form

$$P = [x_1 \colon ... \colon x_{i-1} \colon 1 \colon x_{i+1} \colon ... \colon x_{n+1}].$$

The coordinates $(x_1, ..., x_{i-1}, x_{i+1}, ..., x_{n+1})$ are called the ***nonhomogeneous coordinates*** *of P with respect to U_i.*

Remark 10.5. Let $\mathbb{P}^n(k)$ be a projective n-space over k. Define $\varphi_i \colon k^n \to U_i$ by

$$\varphi_i(x_1, ..., x_n) = [x_1 \colon ... \colon x_{i-1} \colon 1 \colon x_{i+1} \colon ... \colon x_n].$$

Then φ_i defines a bijective correspondence between k^n and U_i. Note that $\mathbb{P}^n(k) = \bigcup_{i=1}^{n+1} U_i$, so that $\mathbb{P}^n(k)$ is covered by $n + 1$ bijective copies of k^n.

Definition 10.6. *Let $\mathbb{P}^n(k)$ be a projective n-space over k. The set*

$$H_\infty = \mathbb{P}^n(k) \setminus U_{n+1} = \{[x_1 \colon ... \colon x_{n+1}] | \, x_{n+1} = 0\}$$

*is called the **hyperplane at infinity**.*

Remark 10.7. Let $\mathbb{P}^n(k)$ be a projective n-space over k. The map $H_\infty \to \mathbb{P}^{n-1}(k)$ given by

$$[x_1: \ldots: x_n: 0] \mapsto [x_1: \ldots: x_n]$$

is bijective. Thus H_∞ may be identified with $\mathbb{P}^{n-1}(k)$ and $\mathbb{P}^n(k) = U_{n+1} \cup H_\infty$ is the union of an affine n-space and a set that gives all directions in affine n-space.

Example 10.8.

1. $\mathbb{P}^0(k)$ is a point.

2. $\mathbb{P}^1(k) = \{[x: 1] |\, x \in k\} \cup \{[1: 0]\}$ is the affine line plus one point at infinity. We call it **projective line** over k.

3. $\mathbb{P}^2(k) = \{[x: y: 1] |\, x, y \in k\} \cup \{[x: y: 0] |\, [x: y] \in \mathbb{P}^1(k)\}$. Here H_∞ is called the **line at infinity.** $\mathbb{P}^2(k)$ is called the **projective plane** over k.

4. Consider a line $\ell: y = ax + b$ in k^2. If we identify k^2 with $U_3 \subseteq \mathbb{P}^2(k)$, the points on the line ℓ correspond to the points $\{[x: y: z] |\, y = ax + bz \text{ and } z \neq 0\} \in \mathbb{P}^2(k)$. Then

   $$\{[x: y: z] |\, y = ax + bz\} \cap H_\infty = \{[1: a: 0]\},$$

   so that all lines with the same slope a, when extended that way, pass through the same point at infinity.

5. Consider the curve C: $y^2 = x^2 + 1$ in k^2. The corresponding set in $\mathbb{P}^2(k)$ is given by the equation $y^2 = x^2 + z^2$, $z \neq 0$. Thus

   $$\{[x: y: z] |\, y^2 = x^2 + z^2\} \cap H_\infty = \{[1: 1: 0], [1: -1: 0]\}.$$

   These are the points where the lines $y = x$ and $y = -x$ intersect the curve.

10.2 Projective algebraic sets.

Definition 10.9. *Let $\mathbb{P}^n(k)$ be a projective n-space over k. A point $P \in \mathbb{P}^n(k)$ is said to be a **zero** of a polynomial $f \in k[x_1, \ldots, x_{n+1}]$ if $f(x_1, \ldots, x_{n+1}) = 0$ for every choice of homogenous coordinated for P; we then write $f(P) = 0$.*

Definition 10.10. *A polynomial $f \in k[x_1, \ldots, x_{n+1}]$ is called a **form** of degree d if it is a sum of monomials of degree d:*

$$f = \sum_{(i_1, \ldots, i_{n+1}) \in S \subseteq \mathbb{N}^{n+1}} a_{i_1 \ldots i_{n+1}} x_1^{i_1} \cdot \ldots \cdot x_{n+1}^{i_{n+1}}, \qquad i_1 + \ldots + i_{n+1} = d.$$

Remark 10.11. A polynomnial $f \in k[x_1, \ldots, x_{n+1}]$ is a form of degree d if and only if

$$f(ab_1, \ldots, ab_{n+1}) = a^d f(b_1, \ldots, b_{n+1}),$$

for all $a, b_1, \ldots, b_{n+1} \in k$.

Proof. If $f = \sum_{(i_1, \ldots, i_{n+1}) \in S \subseteq \mathbb{N}^{n+1}} a_{i_1 \ldots i_{n+1}} x_1^{i_1} \cdot \ldots \cdot x_{n+1}^{i_{n+1}}$ where $i_1 + \ldots + i_{n+1} = d$ for all $(i_1, \ldots, i_n) \in S$, then for any $a, b_1, \ldots, b_{n+1} \in k$:

$$\begin{aligned}
f(ab_1, \ldots, ab_{n+1}) &= \sum_{(i_1, \ldots, i_{n+1}) \in S \subseteq \mathbb{N}^{n+1}} a_{i_1 \ldots i_{n+1}} (ab_1)^{i_1} \cdot \ldots \cdot (ab_{n+1})^{i_{n+1}} \\
&= \sum_{(i_1, \ldots, i_{n+1}) \in S \subseteq \mathbb{N}^{n+1}} a_{i_1 \ldots i_{n+1}} a^{i_1 + \cdots + i_{n+1}} {b_1}^{i_1} \cdot \ldots \cdot {b_{n+1}}^{i_{n+1}} \\
&= a^d \sum_{(i_1, \ldots, i_{n+1}) \in S \subseteq \mathbb{N}^{n+1}} a_{i_1 \ldots i_{n+1}} {b_1}^{i_1} \cdot \ldots \cdot {b_{n+1}}^{i_{n+1}} \\
&= a^d f(b_1, \ldots, b_{n+1}).
\end{aligned}$$

Conversely, assume that $f(ab_1, \ldots, ab_{n+1}) = a^d f(b_1, \ldots, b_{n+1})$, for all $a, b_1, \ldots, b_{n+1} \in k$. Let $f = \sum_{(i_1, \ldots, i_{n+1}) \in S \subseteq \mathbb{N}^{n+1}} a_{i_1 \ldots i_{n+1}} x_1^{i_1} \cdot \ldots \cdot x_{n+1}^{i_{n+1}}$. Then

$$f(ab_1, \ldots, ab_{n+1}) = \sum_{(i_1, \ldots, i_{n+1}) \in S \subseteq \mathbb{N}^{n+1}} a_{i_1 \ldots i_{n+1}} a^{i_1 + \cdots + i_{n+1}} {b_1}^{i_1} \cdot \ldots \cdot {b_{n+1}}^{i_{n+1}}.$$

On the other hand

$$a^d f(b_1, \ldots, b_{n+1}) = \sum_{(i_1, \ldots, i_{n+1}) \in S \subseteq \mathbb{N}^{n+1}} a_{i_1 \ldots i_{n+1}} a^d {b_1}^{i_1} \cdot \ldots \cdot {b_{n+1}}^{i_{n+1}},$$

which yields $a^{i_1 + \cdots + i_{n+1}} = a^d$, and, consequently, $i_1 + \ldots + i_{n+1} = d$ for all $(i_1, \ldots, i_n) \in S$. $\square$

Remark 10.12. Let $\mathbb{P}^n(k)$ be a projective n-space over k, let $P \in \mathbb{P}^n(k)$. If $f \in k[x_1, \ldots, x_{n+1}]$ is a form of degree d and f vanishes at one representative of P, then it vanishes at every representative of P.

Definition 10.13. *Let $\mathbb{P}^n(k)$ be a projective n-space over k. A **projective algebraic set** V is a subset of the projective n-space $\mathbb{P}^n(k)$ consisting of all common zeros of some set of forms $\mathcal{S} \subseteq k[x_1, \ldots, x_{n+1}]$:*

$$V = \{[a_1 : \ldots : a_{n+1}] \in \mathbb{P}^n(k) |\ f(a_1, \ldots, a_{n+1}) = 0 \text{ for all } f \in \mathcal{S}\}.$$

We shall call the set V to be defined by the set of forms $\mathcal{S}$ and denote by $V = \mathcal{Z}(\mathcal{S})$.

Definition 10.14. *An ideal $\mathfrak{a} \lhd k[x_1, \ldots x_{n+1}]$ is called **homogeneous** if for every $f \in \mathfrak{a}$, if*

$$f = \sum_{d=0}^{m} f^{(d)},$$

where $f^{(d)}$ is a form of degree d, then also $f^{(0)}, \ldots, f^{(m)} \in \mathfrak{a}$.

Proposition 10.15. *An ideal $\mathfrak{a} \lhd k[x_1, \ldots, x_{n+1}]$ is homogeneous if and only if it is generated by a finite set of forms.*

Proof. Assume that $\mathfrak{a} \lhd k[x_1, \ldots, x_{n+1}]$ is homogeneous. Since $k[x_1, \ldots, x_{n+1}]$ is Noetherian, $\mathfrak{a} = \langle f_1, \ldots, f_k \rangle$, for some $f_1, \ldots, f_k \in k[x_1, \ldots, x_{n+1}]$. Write $f_i = \sum_{d=0}^{m_i} f_i^{(d)}$, where $f_i^{(d)}$ is a form of degree d, $i \in \{1, \ldots, k\}$. Then $f_i^{(d)} \in \mathfrak{a}$, as $\mathfrak{a}$ is homogeneous, and hence $\left\langle f_i^{(d)} | \, i \in \{1, \ldots, k\}, d \in \{0, \ldots, \max\{m_1, \ldots, m_k\}\} \right\rangle \subseteq \mathfrak{a}$. But as every element of $\mathfrak{a}$ is a combination of $f_1, \ldots, f_k$ which, in turn, are combinations of $f_i^{(d)} | \, i \in \{1, \ldots, k\}, d \in \{0, \ldots, \max\{m_1, \ldots, m_k\}\}$, the other inclusion also holds.

Conversely, assume that $\mathfrak{a} = \langle f_1^{(d_1)}, ..., f_k^{(d_k)} \rangle$, where $f_i^{(d_i)} \in k[x_1, ..., x_{n+1}]$ is a form of degree d_i. Let $f \in \mathfrak{a}$. Write $f = \sum_{i=m}^{r} f_i$, where $\deg f_i = i$. It suffices to show that $f_m \in \mathfrak{a}$, for then $f - f_m \in \mathfrak{a}$ and an inductive argument finishes the proof. Write $f = \sum_{i=1}^{k} a_i \cdot f_i^{(d_i)}$, for some $a_1, ..., a_k \in k[x_1, ..., x_{n+1}]$. Comparing terms with the same degree we conclude that $f_m = \sum_{i \in \{i | d_i = m\}} a_i \cdot f_i^{(d_i)}$, so $f_m \in \mathfrak{a}$. □

Remark 10.16. Let $\mathcal{S} \subseteq k[x_1, ..., x_{n+1}]$ be a set of forms and let $\mathfrak{a}$ be the homogenous ideal of $k[x_1, ..., x_{n+1}]$ generated by $\mathcal{S}$. Then

$$\mathcal{Z}(\mathcal{S}) = \mathcal{Z}(\mathfrak{a}).$$

Remark 10.17. Let $\mathcal{S} \subseteq k[x_1, ..., x_{n+1}]$ be a set of forms. Then there exists a finite set of forms $\{f_1, ..., f_r\} \subseteq k[x_1, ..., x_{n+1}]$ such that

$$\mathcal{Z}(\mathcal{S}) = \mathcal{Z}(f_1, ..., f_r).$$

Remark 10.18. Let $\mathbb{P}^n(k)$ be a projective n-space over k, let $V \subseteq \mathbb{P}^n(k)$ be a projective algebraic set. The set $\mathcal{I}(V)$ of all polynomials whose common zeros coincide with V:

$$\mathcal{I}(V) = \{f \in k[x_1, ..., x_{n+1}] | \, f(a_1, ..., a_{n+1}) = 0 \text{ for all } [a_1 \colon ... \colon a_{n+1}] \in V\}$$

is a homogenous ideal of $k[x_1, ..., x_n]$.

Definition 10.19. *Let $\mathbb{P}^n(k)$ be a projective n-space over k, let $V \subseteq \mathbb{P}^n(k)$ be a projective algebraic set. The ideal $\mathcal{I}(V)$ consisting of polynomials whose common zeros constitute V shall be called the **ideal of the projective algebraic set** V.*

Remark 10.20. Let $\mathbb{P}^n(k)$ be a projective n-space over k, let $V, V_1, V_2 \subseteq \mathbb{P}^n(k)$ be projective algebraic sets, let $\mathfrak{a}, \mathfrak{a}_1, \mathfrak{a}_2$ be homogenous ideals of $k[x_1, ..., x_{n+1}]$. Then:

1. $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \Rightarrow \mathcal{Z}(\mathfrak{a}_1) \supseteq \mathcal{Z}(\mathfrak{a}_2)$,
2. $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) \supseteq \mathfrak{a}$,
3. $\mathcal{Z}(\mathcal{I}(V)) = V$,
4. $V_1 \subseteq V_2 \Leftrightarrow \mathcal{I}(V_1) \supseteq \mathcal{I}(V_2)$,
5. $V_1 = V_2 \Leftrightarrow \mathcal{I}(V_1) = \mathcal{I}(V_2)$.

10.3 Projective algebraic varietes.

Definition 10.21. *Let $\mathbb{P}^n(k)$ be a projective n-space over k. A nonempty projective algebraic set $V \subseteq \mathbb{P}^n(k)$ will be called a **projective algebraic variety** if the homogenous ideal $\mathcal{I}(V)$ of the ring $k[x_1, ..., x_{n+1}]$ is prime.*

Definition 10.22. *Let $\mathbb{P}^n(k)$ be a projective n-space over k. A nonempty projective algebraic set $V \subseteq \mathbb{P}^n(k)$ will be called **irreducible**, if for projective algebraic sets $A, B \subseteq \mathbb{P}^n(k)$:*

$$V = A \cup B \Rightarrow V = A \vee V = B.$$

Theorem 10.23. *Let $\mathbb{P}^n(k)$ be a projective n-space over k. A nonempty projective algebraic set $V \subseteq \mathbb{P}^n(k)$ is irreducible if and only if it is a projective algebraic variety.*

Theorem 10.24. *Let $\mathbb{P}^n(k)$ be a projective n-space over k. Every projective algebraic set V is a finite sum of projective algebraic varieties:*

$$V = V_1 \cup \ldots \cup V_r, \quad r \geq 1.$$

If in the above decomposition the varieties V_i are incomparable (that is $V_i \not\subset V_j$ for $i \neq j$), then they are uniquely defined.

10.4 Projective Nullstellensatz.

Definition 10.25. *Let $\mathbb{P}^n(k)$ be a projective n-space over k, let $V \subseteq \mathbb{P}^n(k)$ be a projective algebraic set. The set*

$$\mathcal{C}(V) = \{(x_1, \ldots, x_{n+1}) \in k^n | \, [x_1 \colon \ldots \colon x_n] \in V\} \cup \{(0, \ldots, 0)\}$$

will be called the ***cone*** *over V.*

Notation 10.26. *To avoid confucion when necessary, we shall write $\mathcal{Z}_a(I)$ and $\mathcal{I}_a(V)$ for affine operations and $\mathcal{Z}_p(I)$ and $\mathcal{I}_p(V)$ for projective operations.*

Remark 10.27. Let $\mathbb{P}^n(k)$ be a projective n-space over k, let $V \subseteq \mathbb{P}^n(k)$ be a nonempty projective algebraic set. Then

$$\mathcal{I}_a(\mathcal{C}(V)) = \mathcal{I}_p(V).$$

Moreover, let $\mathfrak{a} \lhd k[x_1, \ldots, x_{n+1}]$ be a homogeneous ideal such that $\mathcal{Z}_p(\mathfrak{a}) \neq \emptyset$. Then

$$\mathcal{C}(\mathcal{Z}_p(\mathfrak{a})) = \mathcal{Z}_a(\mathfrak{a}).$$

Corollary 10.28. (projective Nullstellensatz) *Let k be algebraically closed, let $\mathfrak{a} \lhd k[x_1, \ldots, x_{n+1}]$ be a homogeneous ideal. Then:*

1. *$\mathcal{Z}_p(\mathfrak{a}) = \emptyset$ if and only if there is an integer N such that $\mathfrak{a}$ contains all forms of degree $\geqslant N$;*
2. *if $\mathcal{Z}_p(\mathfrak{a}) \neq \emptyset$, then $\mathcal{I}_p(\mathcal{Z}_p(\mathfrak{a})) = \mathrm{rad}(\mathfrak{a})$.*

Proof.

1. The following four consitions are equivalent:

   i. $\mathcal{Z}_p(\mathfrak{a}) = \emptyset$,

   ii. $\mathcal{Z}_p(\mathfrak{a}) \subseteq \{(0, \ldots, 0)\}$,

   iii. $\mathrm{rad}(\mathfrak{a}) = \mathcal{I}_a(\mathcal{Z}_a(\mathfrak{a})) \supseteq \langle x_1, \ldots, x_{n+1} \rangle$ (by the affine Nullstellensatz),

   iv. $\langle x_1, \ldots, x_{n+1} \rangle^N \subseteq \mathfrak{a}$.

2. $\mathcal{I}_p(\mathcal{Z}_p(\mathfrak{a})) = \mathcal{I}_a(\mathcal{C}(\mathcal{Z}_p(\mathfrak{a}))) = \mathcal{I}_a(\mathcal{Z}_a(\mathfrak{a})) = \mathrm{rad}(\mathfrak{a})$. □

10.5 Zariski topology.

Lemma 10.29. *A finite sum of profective algebraic sets is a projective algebraic set. To be more precise, let $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ be homogenous ideals of the ring $k[x_1, \ldots, x_{n+1}]$. Then*

$$\mathcal{Z}(\mathfrak{a}_1) \cup \ldots \cup \mathcal{Z}(\mathfrak{a}_m) = \mathcal{Z}(\mathfrak{a}_1 \cdot \ldots \cdot \mathfrak{a}_m),$$

where $\mathfrak{a}_1 \cdot \ldots \cdot \mathfrak{a}_m = \{\sum_{i=1}^{k} a_{i1} a_{i2} \ldots a_{im} | \, k \in \mathbb{N}, a_{ij} \in \mathfrak{a}_j, j \in \{1, \ldots, m\}, i \in \{1, \ldots, k\}\}$.

Remark 10.30. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ be homogenous ideals of the ring $k[x_1, \ldots, x_{n+1}]$. Then

$$\mathcal{Z}(\mathfrak{a}_1 \cdot \ldots \cdot \mathfrak{a}_m) = \mathcal{Z}(\mathfrak{a}_1 \cap \ldots \cap \mathfrak{a}_m).$$

Lemma 10.31. *Intersection of any number of projective algebraic sets is a projective algebraic set. To be more precise, let* $\{\mathfrak{a}_i | \, i \in I\}$ *be a family of homogenous ideals of the ring* $k[x_1, \ldots, x_{n+1}]$. *Then*

$$\bigcap_{i \in I} \mathcal{Z}(\mathfrak{a}_i) = \mathcal{Z}\left(\left(\bigcup_{i \in I} \mathfrak{a}_i\right)\right).$$

Remark 10.32. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ be homogenous ideals of the ring $k[x_1, \ldots, x_{n+1}]$. Then

$$\mathcal{Z}(\mathfrak{a}_1 + \ldots + \mathfrak{a}_m) = \mathcal{Z}(\langle \mathfrak{a}_1 \cup \ldots \cup \mathfrak{a}_m \rangle).$$

Theorem 10.33. *In* $\mathbb{P}^n(k)$ *there is a topology whose closed sets are projective algebraic sets in* $\mathbb{P}^n(k)$.

Definition 10.34. *The topology of* $\mathbb{P}^n(k)$ *defined by projective algebraic sets is called the* ***Zariski topology*** *in* $\mathbb{P}^n(k)$.