9 Rational maps of affine algebraic sets. Birational equivalence of affine algebraic sets.

Definition 9.1. *Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes. A **rational map** $f: V \to W$ is a map such that there exist $f_1, \ldots, f_m \in k(V)$ such that $f(a) = (f_1(a), \ldots, f_m(a))$, for all the points $a \in V$ where all the rational functions $f_1, \ldots, f_m \in k(V)$ are defined.*

Remark 9.2. Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes, let $f: V \to W$ be a rational map, $f = (f_1, \ldots, f_m)$ with $f_1, \ldots, f_m \in k(V)$. There exists an open set $\emptyset \neq U \subseteq V$ such that $f_1 \restriction_U, \ldots, f_m \restriction_U$ are regular on U. In other words, we can think of rational maps as defined on open subsets.

Proof. By Remark 8.6 there exist nonempty open subsets $U_1, \ldots, U_m \subseteq V$ such that $f_i \restriction_{U_i}$ is regular on U_i. The set $U = U_1 \cap \ldots \cap U_m$ is open, as a finite intersection of open sets, and it suffices to show that it is nonempty.

Indeed, suppose that $\bigcap_{i=1}^m U_i = \emptyset$. Say $U_i = V \setminus V_i$, for some closed subset V_i, $i \in \{1, \ldots, m\}$. But then $V_i \neq V$ and $V = \bigcup_{i=1}^m V_i$ contradicting the fact that V, as a variety, is irreducible. □

Remark 9.3. Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes, let $f_1, \ldots, f_m \in k(V)$. Then $f_1, \ldots, f_m$ define a rational map $f: V \to W$.

Proof. It suffices to check that at every point $a \in V$ where all the rational functions $f_1, \ldots, f_m$ are defined we have, in fact, $(f_1(a), \ldots, f_m(a)) \in W$. Let $U \subseteq V$ be the nonempty open set such that $f_1 \restriction_U, \ldots, f_m \restriction_U$ are regular on U. Let $u \in \mathcal{I}(W)$. Then $u(f_1, \ldots, f_m) \in k(V)$ and $u(f_1, \ldots, f_m)$ vanishes at every point of U. As a nonempty open set in V, U is dense in V by Remark 5.5. $u(f_1, \ldots, f_m)$ is continuous and vanishes on the dense set U, so it vanishes on V. Since $u \in \mathcal{I}(W)$ was chosen arbitrarily, this yields $(f_1(a), \ldots, f_m(a)) \in W$. □

Remark 9.4. Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes, let $f: V \to W$ be a rational map and assume that $f(V)$ is dense in W. The map f defines a field embedding $f^*: k(W) \to k(V)$.

Proof. Let $f_1, \ldots, f_m \in k(V)$ be such that $f = (f_1, \ldots, f_m)$ and let $U \subseteq V$ be the nonempty open set such that $f_1 \restriction_U, \ldots, f_m \restriction_U$ are regular on U. Consider f as a map $f: U \to f(V)$. For $\varphi \in k[W]$, $\varphi = \Phi + \mathcal{I}(W)$, $\Phi \in k[x_1, \ldots, x_m]$, define $f^*(\varphi) = \Phi(f_1, \ldots, f_m)$. Clearly $\Phi(f_1, \ldots, f_m) \in k[V] \subseteq k(V)$, so that $f^*: k[W] \to k(V)$ is a homomorphism, and it suffices to check that it is injective.

If $f^*(\varphi) = 0$ for $\varphi = \Phi + \mathcal{I}(W) \in k[W]$, then $\Phi = 0$ on $f(V)$. But if $\Phi \neq 0$ on W, then the equality $\Phi = 0$ defines a closed subset W' of W. Then $\varphi(V) \subseteq W'$, but this contradicts the assumption that $f(V)$ is dense in W.

The embedding $\varphi^*: k[W] \to k(V)$ can be extended in an obvious way to $\varphi^*: k(W) \to k(V)$. □

Definition 9.5. *Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes, let $f: V \to W$ be a rational map such that $f(V)$ is dense in W. The map f is a **birational equivalence** if there is a rational map g: $W \to V$ such that $g(W)$ is dense in V and*

$$f \circ g = 1_W \text{ and } g \circ f = 1_V.$$

*In this case we say that V and W are **birationally equivalent** or **birational**.*

Corollary 9.6. *Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes. Then V and W are birationally equivalent if and only if $k(V) \cong k(W)$.*

Proof. If V and W are birationally equivalent, then $k(V) \cong k(W)$ by Remark 9.4. Conversely, if $k(V) \cong k(W)$, then the rational functions $x_i = X_i + \mathcal{I}(V)$ correspond to rational functions $f_i \in k(W)$. One checks that $f = (f_1, \ldots, f_m)$ is a birational equivalence. □

Example 9.7. Let $V = \mathcal{Z}(xy-1)$ and $W = \mathcal{Z}(y)$, let $f: V \to W$ be given by $(x,y) \mapsto (x,0)$. This is a birational equivalence, but not an isomorphism.

Example 9.8. Let $V = \mathcal{Z}(y)$ and $W = \mathcal{Z}(y^2 - x^3)$, let $f: V \to W$ be given by $(x,0) \mapsto (x^2, x^3)$. This is a birational equivalence (the inverse map $g: W \to V$ being $(x,y) \mapsto \left(\frac{y}{x}, 0\right)$), but not an isomorphism.

Proposition 9.9. *Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine varietes, let f: $V \to W$ be a birational equivalence. Then there exist open subsets $U \subseteq V$ and $U' \subseteq W$ which are isomorphic.*

Proof. Let $g: W \to V$ be the birational map such that $f \circ g = 1_W$ and $g \circ f = 1_V$. Let $U_1 \subseteq V$ be the open set on which f is defined, and, likewise, $U_2 \subseteq W$ the open set where g is defined. Then $f \circ g$ is the identity map on $U_2 \cap g^{-1}(U_1)$ and $g \circ f$ is the identity map on $U_1 \cap f^{-1}(U_2)$. Thus f and g define an isomorphism between $U = f^{-1} \circ g^{-1}(U_1)$ and $U' = g^{-1} \circ f^{-1}(U_2)$. □

Proposition 9.10. (Noether normalization lemma) *Let k be algebraically closed, and $k \subseteq K$ a finitely generated field extension. Then there exist elements $z_1, ..., z_{d+1} \in K$ with $K = k(z_1, ..., z_{d+1})$ such that $z_1, ..., z_d$ are algebraically independent over k, and z_{d+1} is separable over $k(z_1, ..., z_{d+1})$.*

Proof. Let K be generated over k by a finite number of elements $t_1, ..., t_n$ and let d be the maximal number of algebraically independent elements among $t_1, ..., t_n$. Changing the order of $t_1, ..., t_n$, if necessary, we might as well assume that $t_1, ..., t_d$ are algebraically independent. Then any element $y \in K$ is algebraically dependent on $t_1, ..., t_d$ and, moreover, there exists a relation $f(t_1, ..., t_d, y) = 0$ with $f(T_1, ..., T_d, T_{d+1})$ irreducible ober k.

Let $f(T_1, ..., T_d, T_{d+1})$ be such a polynomial for $t_1, ..., t_d, t_{d+1}$. We claim that the partial derivative $\frac{\partial f}{\partial T_i}(T_1, ..., T_d, T_{d+1}) \neq 0$ for at least one $i \in \{1, ..., d+1\}$. Indeed, if this was not the case, then each T_i occurs in f in powers that are multiples of the characteristic p of the field k, that is, f is of the form $f = \sum a_{i_1 \dots i_{d+1}} T_1^{p i_1} \cdot \ldots \cdot T_{d+1}^{p i_{d+1}}$. Set $a_{i_1 \dots i_{d+1}} = b_{i_1 \dots i_{d+1}}^p$ and $g = f = \sum b_{i_1 \dots i_{d+1}} T_1^{i_1} \cdot \ldots \cdot T_{d+1}^{i_{d+1}}$. Then we get $f = g^p$, which contradicts the irreducibility of f.

If $\frac{\partial f}{\partial T_i}(T_1, ..., T_d, T_{d+1}) \neq 0$, the d elements $t_1, ..., t_{i-1}, t_{i+1}, ..., t_{d+1}$ are algebraically independent over k. Indeed, t_i is algebraically independent over $k(t_1, ..., t_{i-1}, t_{i+1}, ..., t_{d+1})$ because $\frac{\partial f}{\partial T_i}(T_1, ..., T_d, T_{d+1}) \neq 0$, so that T_i occurs in f. Thus if $t_1, ..., t_{i-1}, t_{i+1}, ..., t_{d+1}$ were algebraically dependent, the transcendence degree of $k(t_1, ..., t_{d+1})$ would be less than d, which contradicts the algebraic independence of $t_1, ..., t_d$.

Thus we can always rearrange $t_1, ..., t_{d+1}$ so that $t_1, ..., t_d$ are algebraically independent over k, and $\frac{\partial f}{\partial T_{d+1}}(T_1, ..., T_d, T_{d+1}) \neq 0$. This shows that t_{d+1} is separable over $k(t_1, ..., t_d)$. Since t_{d+2} is algebraic over $k(t_1, ..., t_d)$, by the Primitive Element Theorem we can find an element $y \in K$ such that $k(t_1, ..., t_{d+2}) = k(t_1, ..., t_d, y)$. Repeating the process of adjoining elements $t_{d+1}, ..., t_n$ we express K as $k(z_1, ..., z_{d+1})$, where $z_1, ..., z_d$ are algebraically independent over k and $f(z_1, ..., z_d, z_{d+1}) = 0$, with f an irreducible polynomial over k with $\frac{\partial f}{\partial T_{d+1}}(T_1, ..., T_d, T_{d+1}) \neq 0$. □

Proposition 9.11. *Let $V \subseteq k^n$ be an affine variety. Then V is birationally equivalent to a hypersurface of some affine space k^m.*

Proof. $k(V)$ is finitely generated over, say $k(V) = k(t_1, ..., t_n)$. We may view $t_1, ..., t_n$ as rational functions on V. Let d be the maximal number of $t_1, ..., t_d$ that are algebraically independent over k. By the Noether Normalization Lemma, $k\ (V)$ can be written in the form $k(z_1, ..., z_{d+1})$, where $z_1, ..., z_d$ are algebraically independent and $f(z_1, ..., z_{d+1}) = 0$ for some irreducible polynomial $f \in k[T_1, ..., T_{d+1}]$ with $\frac{\partial f}{\partial T_{d+1}}(T_1, ..., T_d, T_{d+1}) \neq 0$. Let $W = \mathcal{Z}(f)$. The function field $k(W)$ of the variety W is obviously isomorphic to $k(V)$, which means that V and W are birationally equivalent. □

A variety is called **rational** if it is birationally equivalent to k^n, for some n.