8 Rational functions field of an affine algebraic variety.

Definition 8.1. *Let $V \subseteq k^n$ be an affine algebraic variety. The field of fractions of the coordinate ring $k[V]$ will be called the* ***field of rational functions*** *of V and denoted by $k(V)$, and its elements* ***rational functions*** *on V.*

Example 8.2. Consider the following easy examples:

- $V = \{(a_1, ..., a_n)\} \in k^n$, $k(V) \cong k$;
- $V = k^k$, $k(V) \cong k(x_1, ..., x_n)$.

Definition 8.3. *Let $V \subseteq k^n$ be an affine algebraic variety. A rational function $\varphi \in k(V)$ is* ***defined*** *at a point $(a_1, ..., a_n) \in V$ if $\varphi = \frac{f}{g}$, for some $f = F + \mathcal{I}(V), g = G + \mathcal{I}(V) \in k[V]$, $F, G \in k[x_1, ..., x_n]$, with $G(a_1, ..., a_n) \neq 0$. In this case we say that $\frac{F(a_1, ..., a_n)}{G(a_1, ..., a_n)} \in k$ is the* ***value*** *of φ at $(a_1, ..., a_n)$, and denote it by $\varphi(a_1, ..., a_n)$.*

Remark 8.4. Let $V \subseteq k^n$ be an affine algebraic variety, let $\varphi \in k(V)$ be defined at $(a_1, ..., a_n) \in V$. The value of φ at $(a_1, ..., a_n)$ is uniquely defined.

Proof. Let $\varphi = \frac{f_1}{g_1} = \frac{f_2}{g_2}$, $f_1 = F_1 + \mathcal{I}(V)$, $f_2 = F_2 + \mathcal{I}(V)$, $g_1 = G_1 + \mathcal{I}(V)$, $g_2 = G_2 + \mathcal{I}(V) \in k[V]$, F_1, $F_2, G_1, G_2 \in k[x_1, ..., x_n]$, with $G_1(a_1, ..., a_n) \neq 0$ and $G_2(a_1, ..., a_n) \neq 0$ be two presentations of φ as a quotient of elements of the coordinate ring of V. Then $f_1 g_2 = f_2 g_1$ in the ring $k[V]$, so that $F_1(a_1, ..., a_n)G_2(a_1, ..., a_n) = F_2(a_1, ..., a_n)G_1(a_1, ..., a_n)$ and thus

$$\frac{F_1(a_1, ..., a_n)}{G_1(a_1, ..., a_n)} = \frac{F_2(a_1, ..., a_n)}{G_2(a_1, ..., a_n)}. \qquad \square$$

Example 8.5. Let $V = \mathcal{Z}(x^2 + y^2 - 1) \subseteq \mathbb{C}^2$. Then $\mathbb{C}(V) \cong \mathbb{C}(x, y)$ with $x^2 + y^2 = 1$. Let $\varphi = \frac{1-y}{x} \in \mathbb{C}(V)$. Then φ is defined at $(0,1) \in V$ and $\varphi(0,1) = 0$, but φ is not defined at $(0, -1)$.

Proof. Since $x^2 = 1 - y^2$ in the ring $\mathbb{C}[V]$, we get

$$\varphi = \frac{1-y}{x} = \frac{(1-y)}{x} \cdot \frac{(1+y)}{(1+y)} = \frac{1-y^2}{x(1+y)} = \frac{x^2}{x(1+y)} = \frac{x}{1+y}$$

and $(1 + y)(0, 1) = 1 \neq 0$, we see that φ is defined at $(0, 1)$ and $\varphi(0, 1) = 0$. On the other hand, suppose that φ is defined at $(0, -1)$, that is

$$\varphi = \frac{1-y}{x} = \frac{f}{g}$$

for some $f = F + \mathcal{I}(V), g = G + \mathcal{I}(V) \in \mathbb{C}[V]$ with $G(0, -1) \neq 0$. Then $(1-y)G(x, y) = xF(x, y)$ in $\mathbb{C}[V]$. But this implies $1 \cdot G(0, -1) = 0 \cdot F(0, -1) = 0$, so that $G(0, -1) = 0$ rendering such a presentation impossible. $\square$

Remark 8.6. Let $V \subseteq k^n$ be an affine algebraic variety. Every rational function $\frac{f}{g} \in k(V)$, $f = F + \mathcal{I}(V)$, $g = G + \mathcal{I}(V) \in k[V]$, $F, G \in k[x_1, ..., x_n]$ determines a function defined on some nonempty open subset $U \subseteq V$ with values in k that we shall also call a rational function.

Proof. Indeed, the set

$$\begin{aligned} U &= \{(a_1, ..., a_n) \in V \mid G(a_1, ..., a_n) \neq 0\} \\ &= V \setminus \{(a_1, ..., a_n) \in V \mid G(a_1, ..., a_n) = 0\} \\ &= V \setminus (V \cap \mathcal{Z}(G)) \end{aligned}$$

is open in the Zariski topology on V induced from k^n. To see that it is nonempty, suppose that $G(a_1, ..., a_n) = 0$ for all $(a_1, ..., a_n) \in V$. But then $G \in \mathcal{I}(V)$, that is $g = 0$ as an element of the coordinate ring $k[V]$, and thus g cannot be a denominator of a quotient in the field of fractions of $k[V]$. $\square$

Remark 8.7. Let $V \subseteq k^n$ be an affine algebraic variety. If the rational functions $\varphi_1, \varphi_2 \in k(V)$ have the same values on a certain nonempty open subset of $U \subseteq V$, then they are equal.

Proof. Say $\varphi_1 = \frac{f_1}{g_1}$ and $\varphi_2 = \frac{f_2}{g_2}$ with $f_1 = F_1 + \mathcal{I}(V)$, $f_2 = F_2 + \mathcal{I}(V)$, $g_1 = G_1 + \mathcal{I}(V)$, $g_2 = G_2 + \mathcal{I}(V) \in k[V]$. If $\varphi_1 = \varphi_2$ on an open subset $U \subseteq V$, then

$$\frac{F_1}{G_1} - \frac{F_2}{G_2} = \frac{F_1G_2 - F_2G_1}{G_1G_2} = 0$$

on U, that is $F_1G_2 - F_2G_1 = 0$ on U as a restriction of a polynomial function $k^n \to k$ to V. Clearly $F_1G_2 - F_2G_1$ is a continuous function in the Zariski topology, and by Remark 5.5 the set U is dense in V, so that $F_1G_2 - F_2G_1 = 0$ on V leading to $\varphi_1 = \varphi_2$ on V. $\square$

Theorem 8.8. *Let $V \subseteq k^n$ be an affine algebraic variety over an algebraically closed field k. If the rational function $\varphi \in k(V)$ is defined at every point of V, then $\varphi \in k[V]$.*

Proof. Since φ is defined at every point of V, then for each such point $\underline{a} \in V$ there exist $f_{\underline{a}} = F_{\underline{a}} + \mathcal{I}(V)$, $g_{\underline{a}} = G_{\underline{a}} + \mathcal{I}(V) \in k[V]$ such that $\varphi = \frac{f_{\underline{a}}}{g_{\underline{a}}}$ with $G_{\underline{a}}(\underline{a}) \neq 0$. Let $\mathfrak{a} = (\{G_{\underline{a}} \mid \underline{a} \in V\}) \lhd k[x_1, ..., x_n]$. Since $k[x_1, ..., x_n]$ is Noetherian, there exists a finite number of points $\underline{a_1}, ..., \underline{a_m} \in V$ such that $\mathfrak{a} = (G_{\underline{a}_1}, ..., G_{\underline{a}_m})$. The polynomials $G_{\underline{a}_1}, ..., G_{\underline{a}_m}$ have no common zero on V, for if $G_{\underline{a}_1}(\underline{a}) = ... = G_{\underline{a}_m}(\underline{a}) = 0$ for some $\underline{a} \in V$, then $G_{\underline{a}}(\underline{a}) \neq 0$ and, as $G_{\underline{a}} \in \mathfrak{a}$, $G_{\underline{a}}(\underline{a}) = P_1(\underline{a})G_{\underline{a}_1}(\underline{a}) + ... + P_m(\underline{a})G_{\underline{a}_m}(a)$, for some $P_1, ..., P_m \in k[x_1, ..., x_n]$, so that $G_{\underline{a}}(\underline{a}) = 0$ – a contradiction. Therefore $\mathcal{Z}(\mathfrak{a} + \mathcal{I}(V)) = \emptyset$, and by Lemma 5.16 there exist $H_1, ..., H_m \in k[x_1, ..., x_n]$ and $Q \in \mathcal{I}(V)$ such that the following equation holds true in the ring $k[x_1, ..., x_n]$:

$$H_1G_{\underline{a}_1} + ... + H_mG_{\underline{a}_m} + Q = 1.$$

But this leads to

$$\begin{aligned} &(H_1 + \mathcal{I}(V))(G_{\underline{a}_1} + \mathcal{I}(V)) + ... + (H_m + \mathcal{I}(V))(G_{\underline{a}_m} + \mathcal{I}(V)) + (Q + \mathcal{I}(V)) = \\ &\qquad = (H_1 + \mathcal{I}(V))(G_{\underline{a}_1} + \mathcal{I}(V)) + ... + (H_m + \mathcal{I}(V))(G_{\underline{a}_m} + \mathcal{I}(V)) \\ &\qquad = 1 + \mathcal{I}(V) \end{aligned}$$

holding true in $k[V]$ and, consequently, $k(V)$. Multiplying both sides by φ and using the fact that $\varphi = \frac{f_{\underline{a}_i}}{g_{\underline{a}_i}}$, $i \in \{1, ..., m\}$, yields:

$$(H_1 + \mathcal{I}(V))\, f_{\underline{a}_1} + ... + (H_m + \mathcal{I}(V))\, f_{\underline{a}_m} = \varphi,$$

that is $\varphi \in k[V]$. $\square$

Definition 8.9. *Let $V \subseteq k^n$ be an affine algebraic set, let $V = V_1 \cup \ldots \cup V_m$ be the decomposition of V into affine algebraic varieties. The **k-algebra of rational functions** of V is defined to be*

$$k(V) = k(V_1) \oplus \ldots \oplus k(V_m)$$

*and its elements are called **rational functions** on V.*

Definition 8.10. *Let $V \subseteq k^n$ be an affine algebraic set. If a rational function $\varphi \in k(V)$ is defined at every point of an open subset $U \subseteq V$, then the restriction $\varphi \upharpoonright_U$ wil be called a **regular** function on U.*

Example 8.11. Let $V = \mathcal{Z}(xy)$. Then $V = \mathcal{Z}(x) \cup \mathcal{Z}(y)$. Let $f = x(y+1)$. Then $f \upharpoonright_{\mathcal{Z}(x)\setminus\{(0,0)\}} = 0$ and $f \upharpoonright_{\mathcal{Z}(y)\setminus\{(0,0)\}} = 1$, $f \in k(V)$, f is regular on both $\mathcal{Z}(x)$ and $\mathcal{Z}(y)$, but not regular on V, as it is not defined on $(0,0)$.

Remark 8.12. Let $V \subseteq k^n$ be an affine algebraic set, let $f \in k(V)$. Then f is continuous on the set of points where it is defined.

Proof. It suffices to check that counterimages of closed sets are closed, which follows directly from the definition of Zariski topology. □

Theorem 8.13. *Let $V \subseteq k^n$ be an affine algebraic variety, let $f = F + \mathcal{I}(V) \in k[V] \setminus \{0\}$, $F \in k[x_1, \ldots, x_n]$, let*

$$k[V]_f = \left\{\varphi \in k(V) \mid \varphi = \frac{h}{f}, m \in \mathbb{Z}, h \in k[V]\right\}$$

and

$$V_f = \{(a_1, \ldots, a_n) \in V \mid F(a_1, \ldots, a_n) \neq 0\}.$$

Then the k-algebra of regular functions on V_f is isomorphic to $k[V]_f$.

Proof. That every rational function from $k[V]_f$ is defined at every point of V_f and thus yields a regular function there – is clear.

Conversely, consider a rational function $\varphi \in k(V)$ regular on V_f. Following the proof of Theorem 8.8, for every $\underline{a} \in V_f$ there exist $h_{\underline{a}} = H_{\underline{a}} + \mathcal{I}(V)$, $f_{\underline{a}} = F_{\underline{a}} + \mathcal{I}(V) \in k[V]$ such that $\varphi = \frac{h_{\underline{a}}}{f_{\underline{a}}}$ with $F_{\underline{a}}(\underline{a}) \neq 0$. Let $\mathfrak{a} = (\{F_{\underline{a}} \mid \underline{a} \in V_f\}) \lhd k[x_1, \ldots, x_n]$. Then $\mathfrak{a} = (F_{\underline{a_1}}, \ldots, F_{\underline{a_m}})$, for some points $\underline{a_1}, \ldots, \underline{a_m} \in V_f$, and the polynomials $F_{\underline{a}_1}$, ..., $F_{\underline{a}_m}$ have no common zeros on V_f i.e. conceivable common zeros of $F_{\underline{a}_1}$, ..., $F_{\underline{a}_m}$ are among zeros of F. Thus $\mathcal{Z}(\mathfrak{a}) \subseteq \mathcal{Z}(F)$, hence $\mathfrak{a} \supseteq (F)$ and there exist G_1, ..., $G_m \in k[x_1, \ldots, x_n]$ such that

$$G_1 F_{\underline{a}_1} + \ldots + G_m F_{\underline{a}_m} = F$$

yielding

$$(G_1 + \mathcal{I}(V)) f_{\underline{a}_1} + \ldots + (G_m + \mathcal{I}(V)) f_{\underline{a}_m} = f,$$

which, after multiplying by φ and using $\varphi = \frac{h_{\underline{a}}}{f_{\underline{a}}}$, $\underline{a} \in V_f$, gives

$$(G_1 + \mathcal{I}(V)) h_{\underline{a}_1} + \ldots + (G_m + \mathcal{I}(V)) h_{\underline{a}_m} = f\varphi,$$

or, denoting by $h = (G_1 + \mathcal{I}(V)) h_{\underline{a}_1} + \ldots + (G_m + \mathcal{I}(V)) h_{\underline{a}_m} \in k[V]$:

$$h = f\varphi,$$

or, equivalently, $\varphi = \frac{h}{f}$. □