7 Coordinate ring of an affine algebraic set. Morphisms of affine algebraic sets. Category of affine algebraic sets.

7.1 Coordinate ring of an affine algebraic set.

Definition 7.1. *Let k be a field, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V. The ring $k[V] := k[x_1, ..., x_n] / \mathcal{I}(V)$ is called the* ***coordinate ring*** *of V.*

Remark 7.2. Let k be a field, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V. Let $f \in k[x_1, ..., x_n]$. The polynomial f defines a polynomial function $k^n \to k$. Let f_V be the restriction of f to the set V, $f_V = f \restriction_V$. Then $f_V = g_V$ if and only if $f + \mathcal{I}(V) = g + \mathcal{I}(V)$.

Proof. Indeed, $f_V = g_V$ means that $f(a_1, ..., a_n) = g(a_1, ..., a_n)$, for all $(a_1, ..., a_n) \in V$, that is $(f - g)(a_1, ..., a_n) = 0$, for all $(a_1, ..., a_n) \in V$, or, equivalently, $f - g \in \mathcal{I}(V)$. □

Remark 7.3. Let k be a field, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V. Let κ: $k[x_1, ..., x_n] \to k[V]$ be the canonical epimorphism, $\kappa(f) = \overline{f} := f + \mathcal{I}(V)$. Then $k[V]$ is a k-ring finitely generated over k by $\overline{x_1}, ..., \overline{x_2}$.

Remark 7.4. Let k be algebraically closed, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V. Then $k[V]$ has no nonzero nilpotents.

Proof. By Hilbert Nullstellensatz, $\mathcal{I}(V)$ is radical, so that, by Lemma 6.7, $k[V] = k[x_1, ..., x_n] / \mathcal{I}(V)$ has no nonzero nilpotents. □

Theorem 7.5. *Let k be algebraically closed. Then a k-ring A is isomorphic to a coordinate ring of an affine algebraic set $V \subseteq k^n$ if and only if it is finitely generated over k and has no nonzero nilpotents.*

Proof. Let $A = k[t_1, ..., t_n]$ be a ring finitely generated over k with no nonzero nilpotents. The map

$$k[x_1, ..., x_n] \to A, \qquad f \mapsto f(t_1, ..., t_n)$$

is a well-defined ring epimorphism. Define by $\mathfrak{a}$ its kernel. The ring $k[x_1, ..., x_n] / \mathfrak{a} \cong A$ has no nonzero nilpotents, hence, by Lemma 6.7, the ideal $\mathfrak{a}$ is radical. Thus $\mathfrak{a} = \mathcal{I}(\mathcal{Z}(\mathfrak{a}))$ and, consequently, $A \cong k[\mathcal{Z}(\mathfrak{a})]$. □

Example 7.6. One easily checks that:

- $V = k^n$, $k[V] \cong k[x_1, ..., x_n]$;
- $V = \emptyset$, $k[V] \cong 0$;
- $V = \{(a_1, ..., a_n)\}$, $k[V] \cong k$.

Example 7.7. Let $V = \mathcal{Z}(f)$, where $f \in k[x_1, ..., x_n]$ is square-free and k is algebraically closed. Then $k[V] \cong k[x_1, ..., x_n] / (f) \cong k[\alpha_1, ..., \alpha_n]$ where $f(\alpha_1, ..., \alpha_n) = 0$.

Proof. By Hilbert Nullstellensatz $\mathcal{I}(V) = \mathcal{I}(\mathcal{Z}(f)) = \operatorname{rad}(f)$. One easily checks that $(f) = \operatorname{rad}(f)$ if and only if f is square-free, which follows that $k[V] \cong k[x_1, ..., x_n] / (f) \cong k[\alpha_1, ..., \alpha_n]$, where $\alpha_i = x_i + (f)$, for $i \in \{1, ..., n\}$. □

Example 7.8. Let $V=\mathcal{Z}(a_1x_1+\ldots+a_nx_n-b)$, where $a_1,\ldots,a_n,b\in k$ and k is algebraically closed. Then $k[V]\cong k[x_1,\ldots,x_{n-1}]$.

Proof. As in the previous example, $k[V]\cong k[x_1,\ldots,x_n]/(a_1x_1+\ldots+a_nx_n-b)\cong k[\alpha_1,\ldots,\alpha_n]$, where $a_1\alpha_1+\ldots+a_n\alpha_n=b$. Relabelling, if necessary, we may assume that $a_n\neq 0$. Further, we may assume that $a_n=1$, since $\mathcal{Z}(a_1x_1+\ldots+a_nx_n-b)=\mathcal{Z}\left(\frac{a_1}{a_n}x_1+\ldots+\frac{a_n}{a_n}x_n-\frac{b}{a_n}\right)$. Thus

$$\alpha_n=b-a_1\alpha_1-\ldots-a_{n-1}\alpha_{n-1},$$

and the ring $k[V]\cong k[\alpha_1,\ldots,\alpha_n]$ is generated by the elements $\alpha_1,\ldots,\alpha_{n-1}$. If suffices to show that these elements are algebraically independent: indeed, if $g(\alpha_1,\ldots,\alpha_{n-1})=0$, for some $g\in k[x_1,\ldots,x_{n-1}]$, then

$$\begin{aligned}\mathcal{I}(V) &= 0_{k[V]}=g(\alpha_1,\ldots,\alpha_{n-1})=g(x_1+\mathcal{I}(V),\ldots,x_{n-1}+\mathcal{I}(V))\\ &= g(x_1,\ldots,x_{n-1})+\mathcal{I}(V),\end{aligned}$$

so that $g\in\mathcal{I}(V)=(h)$, that is h divides g in the ring $k[x_1,\ldots,x_n]$. But this is impossible, since x_n appears in h with a nonzero coefficient, and does not appear in any of the monomials of g. Thus $\alpha_1,\ldots,\alpha_{n-1}$ are algebraically independent over k, and thus $k[\alpha_1,\ldots,\alpha_{n-1}]\cong k[x_1,\ldots,x_{n-1}]$. □

7.2 Basic notions from category theory.

Definition 7.9. *A **category** $\mathcal{C}$ consists of a class of **objects** $\mathrm{Ob}(\mathcal{C})$, denoted by $A,B,C,\ldots$ and a class of **morphisms** (or **arrows**) $\mathrm{Ar}(\mathcal{C})$ together with:*

1. *classes of pairwise disjoint arrows $\mathrm{Hom}_{\mathcal{C}}(A,B)$, one for each pair of objects $A,B\in\mathrm{Ob}(\mathcal{C})$; and elements f of the class $\mathrm{Hom}_{\mathcal{C}}(A,B)$ shall be called a **morphism** from A to B and denoted by $A\xrightarrow{f}B$ or $f\colon A\to B$,*
2. *functions $\mathrm{Hom}_{\mathcal{C}}(B,C)\times\mathrm{Hom}_{\mathcal{C}}(A,B)\to\mathrm{Hom}_{\mathcal{C}}(A,C)$, for each triple of objects A, B, $C\in\mathrm{Ob}(\mathcal{C})$, called **composition** of morphisms; for morphisms $A\xrightarrow{f}B$ and $B\xrightarrow{g}C$ values of this function shall be denoted by $(g,f)\mapsto g\circ f$, and the morphism $A\xrightarrow{g\circ f}C$ shall be called the composition of morphisms $A\xrightarrow{f}B$ and $B\xrightarrow{g}C$.*

Moreover, we require that the following two axioms hold true:

***Associativity.** If $A\xrightarrow{f}B$, $B\xrightarrow{g}C$ and $C\xrightarrow{h}D$ are morphisms in $\mathcal{C}$, then*

$$h\circ(g\circ f)=(h\circ g)\circ f.$$

***Identity.** For every object $B\in\mathrm{Ob}(\mathcal{C})$ there exists a morphism $B\xrightarrow{1_B}B$ such that for all morphisms $A\xrightarrow{f}B$ and $B\xrightarrow{g}C$*

$$1_B\circ f=f \text{ and } g\circ 1_B=g.$$

*If the classes $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Ar}(\mathcal{C})$ are sets, we shall call the category $\mathcal{C}$ **small**. If all classes $\mathrm{Hom}(A,B)$ are sets, we shall call the category $\mathcal{C}$ **locally small.***

Example 7.10.

1. We shall set the notation for a number of familiar categories here:
   - $\mathcal{S}$et is the category of sets with functions as morphisms;

- $\mathcal{G}$rp is the category of groups with group homomorphisms as morphisms;
- $\mathcal{T}$op is the category of topological spaces with continuous functions as morphisms.
- $\mathcal{A}b$ is the category of Abelian groups;
- $\mathcal{R}$ng is the category of rigns;
- $\mathcal{F}$ield is the category of fields;
- $k-\mathcal{V}$ect is the category of $k-$vector spaces;
- $k-\mathcal{A}\mathrm{lg}^0_{\mathrm{fg}}$ is the category of finitely generated k-algebras over an algebraically closed field k with no nonzero nilpotent elements.

All these categories are locally small, but not small.

2. The notion of a category allows for a different take on familiar constructions in mathematics. For example, consider a partial order $(P, \leqslant)$. One checks that considering the elements of P as objects, and defining morphisms by

$$a \rightarrow b \qquad \Leftrightarrow \qquad a \leqslant b$$

one obains a category, which is small provided P is a set.

Definition 7.11. *Let $\mathcal{C}$ be a category. If, for two objects A, $B \in \mathrm{Ob}(\mathcal{C})$ there exist morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$ such that*

$$f \circ g = 1_B \text{ and } g \circ f = 1_A$$

then we say that objects A and B are ***isomorphic*** *and write $A \cong B$.*

Definition 7.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A* ***covariant functor*** *F from $\mathcal{C}$ to $\mathcal{D}$ is a pair of maps $\mathrm{Ob}(\mathcal{C}) \rightarrow \mathrm{Ob}(\mathcal{D})$ and $\mathrm{Ar}(\mathcal{C}) \rightarrow \mathrm{Ar}(\mathcal{D})$ (denoted by the same symbol F), that assign to each object $A \in \mathrm{Ob}(\mathcal{C})$ an object $F(A) \in \mathrm{Ob}(\mathcal{D})$ and to each morphism $A \xrightarrow{f} B$ in $\mathrm{Ar}(\mathcal{C})$ a morphism $F(A) \xrightarrow{F(f)} F(B)$ in $\mathrm{Ar}(\mathcal{D})$ in a way that the following two axioms are satisfied:*

1. *$F(1_A) = 1_{F(A)}$, for every object $A \in \mathrm{Ob}(\mathcal{C})$;*
2. *$F(g \circ f) = F(g) \circ F(f)$, for all arrows $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ in $\mathrm{Ar}(\mathcal{C})$.*

A ***contravariant functor*** *is defined in an analogous way, but to each morphism $A \xrightarrow{f} B$ in $\mathrm{Ar}(\mathcal{C})$ a morphism $F(B) \xrightarrow{F(f)} F(A)$ in $\mathrm{Ar}(\mathcal{D})$ is assigned and the axiom 2. is replaced with:*

2'. *$F(g \circ f) = F(f) \circ F(g)$, for all arrow $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ in $\mathrm{Ar}(\mathcal{C})$.*

Example 7.13.

1. **Identity functors.** For every category $\mathcal{C}$ the map $I_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}$ given by

$$I_{\mathcal{C}}(A) = A, \text{ for every object } A \in \mathrm{Ob}(\mathcal{C}), \qquad I_{\mathcal{C}}(f) = f, \text{ for every morphism } A \xrightarrow{f} B \text{ in } \mathcal{C}$$

is a covariant functor that shall be called the identity functor.

2. **Forgetful functors.** The map $F\colon \mathcal{G}\mathrm{rp} \to \mathcal{S}\mathrm{et}$ given by

   $F(G) = G$, for every object $G \in \mathrm{Ob}(\mathcal{G}\mathrm{rp})$, $\qquad F(f) = f$, for every morphism $G \xrightarrow{f} H$ in $\mathcal{G}\mathrm{rp}$

   is a covariant functor that shall be called the forgetful functor. In the same way we can define forgetful functors $\mathcal{R}\mathrm{ng} \to \mathcal{S}\mathrm{et}$, $\mathcal{T}\mathrm{op} \to \mathcal{S}\mathrm{et}$ etc.

3. **Free functors.** The map $F\colon \mathcal{S}\mathrm{et} \to \mathcal{A}b$ given by

   $$F(X) = \text{free Abelian group with basis } X, \text{ for every object } X \in \mathrm{Ob}(\mathcal{A}b),$$

   and

   $F(f) =$ the uniquely defined morphism $\overline{f}$ s.t. $\overline{f}\restriction_X = f$, for every morphism $X \xrightarrow{f} Y$ in $\mathcal{S}\mathrm{et}$

   is a covariant functor that creates free Abelian groups. In the same way we can define free fuctors $\mathcal{S}\mathrm{et} \to \mathcal{G}\mathrm{rp}$ etc.

4. For a category $\mathcal{C}$ we define the **opposite category** $\mathcal{C}^{\mathrm{op}}$ as follows: $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$, and for $A, B \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$

   $$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(B, A)$$

   and

   $$f^{\mathrm{op}} \circ g^{\mathrm{op}} = (g \circ f)^{\mathrm{op}}.$$

   For example if $\mathcal{C}$ consists of the following objects and morphisms:

   $$A \to B \to C \to D,$$

   then $\mathcal{C}^{\mathrm{op}}$ is of the following form:

   $$A \leftarrow B \leftarrow C \leftarrow D.$$

   If $F\colon \mathcal{C} \to \mathcal{D}$ is a contravariant functor, then $\bar{F}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ defined by

   $\bar{F}(A) = F(A)$, for every object $A \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$, $\bar{F}(f^{\mathrm{op}}) = F(f)$, for every morphism $A \xrightarrow{f} B$ in $\mathcal{C}$

   is a covariant functor.

Definition 7.14. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A covariant functor $F\colon \mathcal{C} \to \mathcal{D}$ is **faithful** if for all objects $A, B \in \mathrm{Ob}(\mathcal{C})$ the induced function*

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$$

*is injective. If, moreover, it is surjective, then F shall be called **fully faithful**.*

Proposition 7.15. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, let $F\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor. Then, for all objects $A, B \in \mathrm{Ob}(\mathcal{C})$, $A \cong B$ if and only if $F(A) \cong F(B)$.*

Proof. Fix two objects $A, B \in \mathrm{Ob}(\mathcal{C})$ and assume that $A \cong B$. Let $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$ be two morphisms such that $f \circ g = 1_B$ and $g \circ f = 1_A$. Then

$$1_{F(B)} = F(1_B) = F(f \circ g) = F(f) \circ F(g) \text{ and } 1_{F(A)} = F(1_A) = F(g \circ f) = F(g) \circ F(f)$$

so that the morphisms $F(A) \xrightarrow{F(f)} F(B)$ and $F(B) \xrightarrow{F(g)} F(A)$ establish the isomorphism $F(A) \cong F(B)$.

Conversely, assume $F(A) \cong F(B)$ and let $F(A) \xrightarrow{\varphi} F(B)$ and $F(B) \xrightarrow{\psi} F(A)$ be two morphisms such that $\varphi \circ \psi = 1_{F(B)}$ and $\psi \circ \varphi = 1_{F(A)}$. Since the maps $\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ and $\mathrm{Hom}_{\mathcal{C}}(B, A) \to \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$ are surjective, there exist morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$ such that $\varphi = F(f)$ and $\psi = F(g)$. Thus

$$1_{F(B)} = \varphi \circ \psi = F(f) \circ F(g) = F(f \circ g) \text{ and } 1_{F(A)} = \psi \circ \varphi = F(g) \circ F(f) = F(g \circ f).$$

On the other hand, $F(1_A) = 1_{F(A)}$ and $F(1_B) = 1_{F(B)}$. Since the maps $\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ and $\mathrm{Hom}_{\mathcal{C}}(B, A) \to \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$ are injective, this yields

$$f \circ g = 1_B \text{ and } g \circ f = 1_A. \qquad \square$$

Definition 7.16. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A covariant functor $F: \mathcal{C} \to \mathcal{D}$ is called an* ***equivalence of categories*** *if it is fully faithful and essentially surjective, that is for every object $B \in \mathrm{Ob}(\mathcal{D})$ there is an object $A \in \mathrm{Ob}(\mathcal{C})$ such that $F(C) = D$.*

7.3 Category of affine algebraic sets.

Definition 7.17. *Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine algebraic sets. A* ***morphism*** *$f: V \to W$ is a map such that there exist $f_1, \ldots, f_m \in k[V]$ such that $f(a) = (f_1(a), \ldots, f_m(a))$, for all $a \in V$.*

Remark 7.18. Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine algebraic sets, let $f_1, \ldots, f_m \in k[V]$. Then $f = (f_1, \ldots, f_m): V \to W$ is a morphism if and only if

$$g(f_1, \ldots, f_m) = 0 \in k[V] \quad \text{for all } g \in \mathcal{I}(W).$$

Proof. Indeed, one easily checks that

$$\begin{aligned}
(f_1(a_1, \ldots, a_n), \ldots, f_m(a_1, \ldots, a_n)) \in W \;&\Leftrightarrow\; g(f_1(a_1, \ldots, a_n), \ldots, f_m(a_1, \ldots, a_n)) = 0 \text{ for all } g \in \mathcal{I}(W)\\
&\Leftrightarrow\; g(f_1, \ldots, f_m)(a_1, \ldots, a_n) = 0 \text{ for all } g \in \mathcal{I}(W)\\
&\Leftrightarrow\; g(f_1, \ldots, f_m) \in \mathcal{I}(V) \text{ for all } g \in \mathcal{I}(W)\\
&\Leftrightarrow\; g(f_1, \ldots, f_m) = 0 \in k[V] \text{ for all } g \in \mathcal{I}(W).
\end{aligned}$$

$\square$

Example 7.19. Consider the following easy examples.

- Let $f \in k[V]$. Then $f: V \to k$ is a morphism.
- Let $f: k^n \to k^m$ be a linear map. Then f is a morphism.
- Let $f: \mathcal{Z}(xy - 1) \to k$ be given by $f(x, y) = x$. Then f is a morphism.
- Let $f: k \to \mathcal{Z}(y^2 - x^3)$ be given by $f(t) = (t^2, t^3)$. Then f is a morphism.

Example 7.20. One easily checks that:

- $\mathcal{Z}(y - x^k) \cong k$ via $f(x, y) = x$ and $g(t) = (t, t^k)$;

- $f\colon \mathcal{Z}(xy-1)\to k$ given by $f(x,y)=x$ is not an isomorphism;
- $f\colon k\to \mathcal{Z}(y^2-x^3)$ given by $f(t)=(t^2,t^3)$ is not an isomorphism, even though it is a bijection.

* We shall denote $k-\mathcal{A}\mathrm{ff}$ the category of affine algebraic sets over an algebraically closed field k with morphisms defined above.

Theorem 7.21. *Let k be algebraically closed and consider the categories $k-\mathcal{A}\mathrm{ff}$ and $k-\mathcal{A}\mathrm{lg}^0_{\mathrm{fg}}$. The assignment*

$$F(V)=k[V]\ \textit{for an affine algebraic set } V\subseteq k^n$$

and

$$F(\varphi)=\varphi^*\ \ \textit{for a morphism of affine algebraic sets}\ \ \varphi\colon V\to W,$$

where $\varphi^*\colon k[W]\to k[V]$ is given by the formula

$$\varphi^*(f)=g\circ\varphi$$

defines an equivalence of categories $k-\mathcal{A}\mathrm{ff}^{\mathrm{op}}$ and $k-\mathcal{A}\mathrm{lg}^0_{\mathrm{fg}}$.

Proof. The map F assigns to an affine algebraic set $V\subseteq k^n$ a finitely generated k-algebra with no nilpotent elements by Theorem 7.5. By the same result F is also essentially surjective. If $\varphi\colon V\to W$ is a morphism between affine algebraic sets V and W, and if $g\in k[W]$, then $g\circ\varphi\in k[V]$. Moreover, the map φ^*: $k[W]\to k[V]$ is a homomorphism of k-algebras. Thus F defines a contravariant functor between categories $k-\mathcal{A}\mathrm{ff}$ and $k-\mathcal{A}\mathrm{lg}^0_{\mathrm{fg}}$, or, equivalently, a covariant functor between the categories $k-\mathcal{A}\mathrm{ff}^{\mathrm{op}}$ and $k-\mathcal{A}\mathrm{lg}^0_{\mathrm{fg}}$. It remains to check that it is fully faithful.

Assume that φ, ψ: $V\to W$ are morphisms of affine algebraic sets and that $\varphi^*=\psi^*$. Say $\varphi=(\varphi_1,\ldots,\varphi_m)$ and $\psi=(\psi_1,\ldots,\psi_m)$ with $\varphi_1,\ldots,\varphi_m,\psi_1,\ldots,\psi_m\in k[V]$. Consider the element $\bar{x}_1=x_1+\mathcal{I}(V)\in k[W]$. Since $\bar{x}_1\circ\varphi=\varphi^*(\bar{x}_1)=\psi^*(\bar{x}_2)=\bar{x}_2\circ\psi$ it follows that $\varphi_1=x_1(\varphi_1,\ldots,\varphi_m)=x_1(\psi_1,\ldots,\psi_m)=\psi_m$. Likewise $\varphi_j=\psi_j$, for $j\in\{2,\ldots,m\}$, so that the map $\mathrm{Hom}_{k-\mathcal{A}\mathrm{ff}}(V,W)\to\mathrm{Hom}_{k-\mathcal{A}\mathrm{lg}^0_{\mathrm{fg}}}(F(V),F(W))$ is injective.

Finally, let f: $k[W]\to k[V]$ be a homomorphism of k-algebras. We shall show that $f=\varphi^*$, for some morphism $\varphi\colon V\to W$. Indeed, consider the elements $\bar{x}_j=x_j+\mathcal{I}(W)$, for $j\in\{1,\ldots,m\}$. Then $\varphi_j=f(\bar{x}_j)\in k[V]$, $j\in\{1,\ldots,m\}$, and consider the map $\varphi=(\varphi_1,\ldots,\varphi_m)\colon V\to k^m$. Clearly $f=\varphi^*$ and all that is left to show is that $\varphi(V)\subseteq W$. Fix $H\in\mathcal{I}(W)$. Then $H(\bar{x}_1,\ldots,\bar{x}_m)=0$ in $k[W]$, hence also $f(H)=0$ on V. Fix $(a_1,\ldots,a_n)\in V$; then $H(\varphi(a_1,\ldots,a_n))=f(H)(a_1,\ldots,a_n)=0$, and therefore $\varphi(a_1,\ldots,a_n)\in W$. ☐