2 Primary decomposition.

2.1 Primary decomposition.

Remark 2.1. Consider the ring $\mathbb{Z}$ and an element $n\in\mathbb{Z}$. Then there exist uniquely determined prime numbers $p_1,\ldots,p_m$ and exponents $k_1,\ldots,k_m\in\mathbb{N}$ such that

$$n=\pm p_1^{k_1}\cdot\ldots\cdot p_m^{k_m}$$

or, equivalently:

$$(n)=(p_1^{k_1})\cdot\ldots\cdot(p_m^{k_m})=(p_1^{k_1})\cap\ldots\cap(p_m^{k_m}).$$

Definition 2.2. *Let R be any ring. An ideal $\mathfrak{q}\lhd R$ is called* ***primary****, if $\mathfrak{q}\neq R$ and for all $a,b\in R$*

$$ab\in\mathfrak{q}\wedge b\notin\mathfrak{q}\Rightarrow\exists n\in\mathbb{N}\quad a^n\in\mathfrak{q}.$$

Example 2.3.

1. Every prime ideal is primary.

2. An ideal in $\mathbb{Z}$ generated by a power of a prime number is primary.

3. Let R be a principal ideal domain. Then $\mathfrak{q}$ is primary if and only if $\mathfrak{q}=\mathfrak{p}^n$, for a prime ideal $\mathfrak{p}$.

Proof. 1. and 2. are obvious. In order to show 3., assume that $\mathfrak{q}=\mathfrak{p}^n$. As R is a PID, it follows that $\mathfrak{p}=(p)$, for some prime element $p\in R$. Consequently, $\mathfrak{q}=(p)^n=(p^n)$. Say $a\cdot b\in\mathfrak{q}=(p^n)$ with $b\notin\mathfrak{q}=(p^n)$, for some $a,b\in R$. Then $p^n|\,a\cdot b$ and $p^n\nmid b$. As a PID, R is a unique factorization domain, so that it follows $p|\,a$, and, consequently, $p^n|\,a$, that is $a^n\in\mathfrak{q}$.

Conversely, assume that $\mathfrak{q}$ is primary. Let $\mathfrak{q}=(c)$, for some $c\in R$. Suppose that $c\neq u\cdot p^n$, for all units $u\in U(R)$, all prime elements $p\in R$, and all $n\in\mathbb{N}$. Then, by unique factorization, c is divisible by two different prime elements, say p and q. Let $c=a\cdot b$ with $p|\,a$ and $q|\,b$. Then $c|\,a\cdot b$ and $c\nmid b$, but also $c\nmid a^n$, for all $n\in\mathbb{N}$, which means that $\mathfrak{q}=(c)$ is not primary − a contradiction. □

Lemma 2.4. *Let R be a ring, let $\mathfrak{q}\lhd R$ be a proper ideal in R. The following conditions are equivalent:*

i. $\mathfrak{q}$ *is primary,*

ii. every zero divisor in $R/\mathfrak{q}$ is nilpotent,

iii. the zero ideal in $R/\mathfrak{q}$ is primary.

Proof. It suffices to notice that $\mathfrak{q}$ being primary is equivalent to the following condition in $R/\mathfrak{q}$:

$$(a+\mathfrak{q})\cdot(b+\mathfrak{q})=\mathfrak{q}\wedge a+\mathfrak{q}\neq\mathfrak{q}\Rightarrow\exists n\in\mathbb{N}\,(a+\mathfrak{q})^n=\mathfrak{q}.$$ □

Example 2.5. The ideal $(x,y^2)\lhd k[x,y]$, where k is any field, is primary, but is not a power of a prime ideal.

Proof. Observe that every polynomial $f(x,y)\in k[x,y]$ can be written as

$$f(x,y)=x\cdot g(x,y)+h(y)=x\cdot g(x,y)+y^2\cdot h_1(y)+a\cdot y+b,$$

with $g(x,y)\in k[x,y]$, $h(y), h_1(y)\in k[y]$ and $a,b\in k$. It then follows that the map

$$k[x,y]/\mathfrak{q}\to k[y]/(y^2), \qquad f(x,y)+\mathfrak{q}\mapsto a\cdot y+b+(y^2)$$

is a well-defined ring isomorphism, so that $k[x,y]/\mathfrak{q}\cong k[y]/(y^2)$.

In order to show that $\mathfrak{q}$ is primary, we note that $k[y]$ is a PID, and y is a prime element of $k[y]$, so that, by Example 2.3.3 the ideal (y^2) is primary. Thus, by Lemma 2.4.iii, the zero ideal in the ring $k[y]/(y^2)$ is primary, and so is the zero ideal in the isomorphic ring $k[x,y]/\mathfrak{q}$, leading to $\mathfrak{q}$ being primary.

We proceed to show that $\mathfrak{q}$ is not a power of a prime ideal. Firstly, $\mathfrak{q}$ is not prime itself, as the ring $k[x,y]/\mathfrak{q}$ is not a domain: the isomorphic ring $k[y]/(y^2)$ has zero divisors, for example $(y+(y^2))^2=(y^2)$. Secondly, suppose that $\mathfrak{q}=\mathfrak{p}^n$, for some prime ideal $\mathfrak{p}\lhd k[x,y]$. Since

$$(x,y^2)=\mathfrak{q}=\mathfrak{p}^n\subseteq\mathfrak{p},$$

it follows that $x,y^2\in\mathfrak{p}$. As $\mathfrak{p}$ is prime, also $y\in\mathfrak{p}$. Consequently, $(x,y)\subseteq\mathfrak{p}$, but as (x,y) is maximal, it follows $(x,y)=\mathfrak{p}$. Thus $\mathfrak{q}=\mathfrak{p}^n=(x,y)^n$. On the other hand

$$(x,y)^2\subsetneq\mathfrak{q}\subsetneq(x,y),$$

which yields a contradiction. □

Definition 2.6. *Let R be a ring. An ideal $\mathfrak{n}\lhd R$, $0\neq\mathfrak{n}$ is **irreducible** if, for all $\mathfrak{a},\mathfrak{b}\lhd R$*

$$\mathfrak{n}=\mathfrak{a}\cap\mathfrak{b}\Rightarrow\mathfrak{n}=\mathfrak{a}\vee\mathfrak{n}=\mathfrak{b}.$$

Example 2.7.

1. Every maximal ideal is irreducible.

2. Every prime ideal is irreducible.

3. An ideal $\mathfrak{n}\lhd R$ is irreducible if and only if the zero ideal in $R/\mathfrak{n}$ is irreducible.

Proof. 1. is obvious. For the proof of 2., suppose that $\mathfrak{p}$ is a prime ideal of a ring R such that $\mathfrak{p}=\mathfrak{a}\cap\mathfrak{b}$, for some $\mathfrak{a},\mathfrak{b}\lhd R$, with $\mathfrak{p}\subsetneq\mathfrak{a}$ and $\mathfrak{p}\subsetneq\mathfrak{b}$. Then there exist $a\in\mathfrak{a}\setminus\mathfrak{p}$ and $b\in\mathfrak{b}\setminus\mathfrak{p}$. Clearly $a\cdot b\in\mathfrak{p}$ implying $a\in\mathfrak{a}$ or $b\in\mathfrak{p}$, which yields a contradiction.

In order to show 3., assume that $\mathfrak{n}$ is an irreducible ideal of a ring R. Let

$$(\mathfrak{n})=\mathfrak{A}\cap\mathfrak{B},$$

for some ideals $\mathfrak{A}$, $\mathfrak{B}\lhd R/\mathfrak{n}$. Let $\mathfrak{a}=\kappa^{-1}(\mathfrak{A})$ and $\mathfrak{b}=\kappa^{-1}(\mathfrak{B})$, where κ denotes the canonical epimorphism $\kappa\colon R\to R/\mathfrak{n}$, $a\overset{\kappa}{\longmapsto}a+\mathfrak{n}$. Then

$$\mathfrak{n}=\kappa^{-1}((\mathfrak{n}))=\kappa^{-1}(\mathfrak{A})\cap\kappa^{-1}(\mathfrak{B})=\mathfrak{a}\cap\mathfrak{b}.$$

As $\mathfrak{n}$ is irreducible, either $\mathfrak{n}=\mathfrak{a}$ or $\mathfrak{n}=\mathfrak{b}$, which leads to $(\mathfrak{n})=\mathfrak{A}$ or $(\mathfrak{n})=\mathfrak{B}$.

Conversely, assume that $(\mathfrak{n})$ is an irreducible ideal in $R/\mathfrak{n}$. Let

$$\mathfrak{n}=\mathfrak{a}\cap\mathfrak{b},$$

for some ideals $\mathfrak{a}, \mathfrak{b} \lhd R$. Let $\mathfrak{A} = \kappa(\mathfrak{a})$ and $\mathfrak{B} = \kappa(\mathfrak{b})$. Then

$$(\mathfrak{n}) = \kappa(\mathfrak{n}) = \kappa(\mathfrak{a} \cap \mathfrak{b}) = \kappa(\mathfrak{a}) \cap \kappa(\mathfrak{b}) = \mathfrak{A} \cap \mathfrak{B};$$

indeed, clearly $\kappa(\mathfrak{a} \cap \mathfrak{b}) \subset \kappa(\mathfrak{a}) \cap \kappa(\mathfrak{b})$, and for the other inclusion fix $\bar{y} \in \kappa(\mathfrak{a}) \cap \kappa(\mathfrak{b})$. Thus $\bar{y} = \kappa(a)$, for some $a \in \mathfrak{a}$, and $\bar{y} = \kappa(b)$, for some $b \in \mathfrak{b}$. Hence $a - b \in \ker \kappa = \mathfrak{n}$, so that $a = b + n$, for some $n \in \mathfrak{n}$, but as $\mathfrak{n} \subseteq \mathfrak{b}$, this yields $a \in \mathfrak{b}$ and, consequently, $a \in \mathfrak{a} \cap \mathfrak{b}$.

Now, by irreducibility of $(\mathfrak{n})$, we either get $(\mathfrak{n}) = \mathfrak{A}$, leading to $\mathfrak{n} = \mathfrak{a}$, or $(\mathfrak{n}) = \mathfrak{B}$, leading to $\mathfrak{n} = \mathfrak{b}$. □

Lemma 2.8. *Let R be Noetherian. Every irreducible ideal in R is primary.*

Proof. Let $\mathfrak{n}$ be an irreducible ideal in a Noetherian ring R. By Lemma 2.4.2 it suffices to show that in the ring $A/\mathfrak{n}$ every zero divisor is nilpotent. Let $\bar{x}, \bar{y} \in R/\mathfrak{n}$ be such that $\bar{x}\bar{y} = \bar{0}$ with $\bar{y} \neq \bar{0}$. For $\bar{t} \in R/\mathfrak{n}$ let

$$\operatorname{Ann} \bar{t} = \{\bar{z} \in R/\mathfrak{n} |\ \bar{z}\bar{t} = 0\}.$$

One easily checks that $\operatorname{Ann} \bar{t} \lhd R/\mathfrak{n}$, and thus

$$\operatorname{Ann} \bar{x} \subseteq \operatorname{Ann} \bar{x}^2 \subseteq \ldots \subseteq \operatorname{Ann} \bar{x}^n \subseteq \ldots$$

is an ascending chain of ideals. Since R is Noetherian, so is $R/\mathfrak{n}$, and hence there exists $n \in \mathbb{N}$ such that

$$\operatorname{Ann} \bar{x}^n = \operatorname{Ann} \bar{x}^{n+1} = \ldots.$$

We claim that $(\bar{x}^n) \cap (\bar{y}) = (\bar{0})$. Indeed, let $\bar{a} \in (\bar{x}^n) \cap (\bar{y})$. Then $\bar{a} = \bar{b}\, \bar{x}^n$ and $\bar{a} = \bar{c}\, \bar{y}$, for some $\bar{b}$, $\bar{c} \in R/\mathfrak{n}$. Hence

$$\bar{b}\bar{x}^{n+1} = \bar{b}\bar{x}^n\bar{x} = \bar{a}\bar{x} = \bar{c}\bar{y}\bar{x} = \bar{c}\bar{0} = \bar{0},$$

so that $\bar{b} \in \operatorname{Ann} \bar{x}^{n+1} = \operatorname{Ann} \bar{x}^n$ and, consequently, $\bar{a} = \bar{b}\bar{x}^n = \bar{0}$. This proves the claim.

By Example 2.7.3 the zero ideal of $R/\mathfrak{n}$ is irreducible. Thus, by the above claim, $(\bar{x}^n) = (\bar{0})$, as $\bar{y} \in (\bar{y})$ and $\bar{y} \neq \bar{0}$. Therefore $\bar{x}^n = \bar{0}$, that is $\bar{x}$ is nilpotent. □

Lemma 2.9. *Let R be Noetherian, let $\mathfrak{a} \lhd R$ be a proper ideal. Then $\mathfrak{a}$ is an intersection of a finite number of irreducible ideals.*

Proof. Suppose that there exists a nonempty family $\mathcal{R}$ of proper ideals that are not intersections of finite numbers of irreducible ideals. Since R is Noetherian, the family $\mathcal{R}$ contains a maximal element $\mathfrak{c}$. In particular, $\mathfrak{c}$ is not irreducible. Let $a, b \in R$ with $a, b \notin \mathfrak{c}$ and let $\mathfrak{a} = \mathfrak{c} + (a)$ and $\mathfrak{b} = \mathfrak{c} + (b)$. Then

$$\mathfrak{c} = \mathfrak{a} \cap \mathfrak{b}, \qquad \mathfrak{c} \subsetneq \mathfrak{a}, \qquad \mathfrak{c} \subsetneq \mathfrak{b},$$

which means that $\mathfrak{a}$, $\mathfrak{b} \notin \mathcal{R}$. Thus both $\mathfrak{a}$ and $\mathfrak{b}$ are intersections of finite numbers of irreducible ideals, and so is $\mathfrak{c}$ – a contradiction. □

Theorem 2.10. *Let R be Noetherian, let $\mathfrak{a} \lhd R$ be a proper ideal. Then $\mathfrak{a}$ is an intersection of a finite number of primary ideals.*

Proof. By Lemma 2.9 every ideal is an intersection of a finite number of irreducible ideals, and by Lemma 2.8 every irreducible ideal in R is primary. □