7 Morphisms of affine algebraic sets. Category of affine algebraic sets. Isomorphisms of affine algebraic sets.

7.1 Morphisms of affine algebraic sets.

Definition 7.1. *Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine algebraic sets. A **morphism** $f: V \to W$ is a map such that there exist $f_1, \ldots, f_m \in k[V]$ such that $f(a) = (f_1(a), \ldots, f_m(a))$, for all $a \in V$.*

Remark 7.2. Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine algebraic sets, let $f_1, \ldots, f_m \in k[V]$. Then $f = (f_1, \ldots, f_m): V \to W$ is a morphism if and only if

$$g(f_1, \ldots, f_m) = 0 \in k[V] \quad \text{for all } g \in \mathcal{I}(W).$$

Example 7.3.

- Let $f \in k[V]$. Then $f\colon V \to k$ is a morphism.
- Let $f\colon k^n \to k^m$ be a linear map. Then f is a morphism.
- Let $f\colon \mathcal{Z}(xy-1) \to k$ be given by $f(x,y) = x$. Then f is a morphism.
- Let $f\colon k \to \mathcal{Z}(y^2 - x^3)$ be given by $f(t) = (t^2, t^3)$. Then f is a morphism.

7.2 Category of affine algebraic sets.

Definition 7.4. *A **category** $\mathcal{C}$ consists of a class of **objects** $\mathrm{Ob}(\mathcal{C})$, denoted by $A, B, C, \ldots$ and a class of **morphisms** (or **arrows**) $\mathrm{Ar}(\mathcal{C})$ together with:*

1. *classes of pairwise disjoint arrows $\mathrm{Hom}(A, B)$, one for each pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$; and elements f of the class $\mathrm{Hom}(A,\, B)$ shall be called a **morphism** from A to B and denoted by $A \xrightarrow{f} B$ or $f\colon A \to B$,*
2. *functions $\mathrm{Hom}(B,\, C) \times \mathrm{Hom}(A,\, B) \to \mathrm{Hom}(A,\, C)$, for each triple of objects $A,\, B,\, C \in \mathrm{Ob}(\mathcal{C})$, called **composition** of morphisms; for morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ values of this function shall be denoted by $(g,\, f) \mapsto g \circ f$, and the morphism $A \xrightarrow{g \circ f} C$ shall be called the composition of morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$.*

Moreover, we require that the following two axioms hold true:

Associativity. *If* $A \xrightarrow{f} B$, $B \xrightarrow{g} C$ *and* $C \xrightarrow{h} D$ *are morphisms in* $\mathcal{C}$, *then*

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Identity. *For every object* $B \in \mathrm{Ob}(\mathcal{C})$ *there exists a morphism* $B \xrightarrow{1_B} B$ *such that for all morphisms* $A \xrightarrow{f} B$ *and* $B \xrightarrow{g} C$

$$1_B \circ f = f \ \ \textit{and} \ g \circ 1_B = g.$$

If the classes $\mathrm{Ob}(\mathcal{C})$ *and* $\mathrm{Ar}(\mathcal{C})$ *are sets, we shall call the category* $\mathcal{C}$ ***small****. If all classes* $\mathrm{Hom}(A, B)$ *are sets, we shall call the category* $\mathcal{C}$ ***locally small.***

Remark 7.5. Affine algebraic sets with morphisms defined in the previous section for a category.

7.3 Isomorphisms of affine algebraic sets.

Definition 7.6. *Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine algebraic sets. A morphism $f: V \to W$ shall be called an* ***isomorphism*** *is there exists a morphism $g: W \to V$ such that*

$$f \circ g = \mathrm{id}_W \quad \textit{and} \quad g \circ f = \mathrm{id}_V.$$

If there exists an isomorphism $f: V \to W$, we shall call the sets V and W ***isomorphic*** *and denote $V \cong W$.*

Example 7.7.

- $\mathcal{Z}(y - x^k) \cong k$ via $f(x, y) = x$ and $g(t) = (t, t^k)$.
- $f\colon \mathcal{Z}(xy - 1) \to k$ given by $f(x, y) = x$ is not an isomorphism.
- $f\colon k \to \mathcal{Z}(y^2 - x^3)$ given by $f(t) = (t^2, t^3)$ is not an isomorphism, even though it is a bijection.