6 Coordinate ring of an affine algebraic set.

6.1 Coordinate ring of an affine algebraic set.

Definition 6.1. *Let k be a field, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V. The ring $k[V] := k[x_1, \ldots, x_n]/\mathcal{I}(V)$ is called the **coordinate ring** of V.*

Remark 6.2. Let k be a field, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V. Let $f \in k[x_1, \ldots, x_n]$. The polynomial f defines a polynomial function $k^n \to k$. Let f_V be the restriction of f to the set V, $f_V = f\restriction_V$. Then $f_V = g_V$ if and only if $f + \mathcal{I}(V) = g + \mathcal{I}(V)$.

Remark 6.3. Let k be a field, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V. Let κ: $k[x_1, ..., x_n] \to k[V]$ be the canonical epimorphism, $\kappa(f) = \overline{f} := f + \mathcal{I}(V)$. Then $k[V]$ is a k-ring finitely generated over k by $\overline{x_1}, ..., \overline{x_2}$.

Remark 6.4. Let k be algebraically closed, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V. Then $k[V]$ has no nonzero nilpotents.

Theorem 6.5. *Let k be algebraically closed. Then a k-ring A is isomorphic to a coordinate ring of an affine algebraic set $V \subseteq k^n$ if and only if it is finitely generated over k and has no nonzero nilpotents.*

Example 6.6.

- $V = k^n$, $k[V] \cong k[x_1, \ldots, x_n]$;
- $V = \emptyset$, $k[V] \cong 0$;
- $V = \{(a_1, \ldots, a_n)\}$, $k[V] \cong k$.

Example 6.7. $V = \mathcal{Z}(f)$, $f \in k[x_1, \ldots, x_n]$ is square-free. $k[V] \cong k[x_1, \ldots, x_n] / (f) \cong k[\alpha_1, \ldots, \alpha_n]$ where $f(\alpha_1, \ldots, \alpha_n) = 0$.

Example 6.8. $V = \mathcal{Z}(a_1x_1 + ... + a_nx_n - b)$, $k[V] \cong k[x_1, ..., x_{n-1}]$.