5 Applications of Nullstellensatz. Maximal ideals in polynomial rings. Radical ideals.

5.1 Decomposition of affine algebraic sets into affine algebraic varieties.

Proposition 5.1. *Let k be algebraically closed, let $\mathfrak{a} \lhd k[x_1, \ldots, x_n]$, and let*

$$\mathfrak{a} = \mathfrak{q}_1 \cdot \ldots \cdot \mathfrak{q}_m = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_m.$$

be the primary decomposition of $\mathfrak{a}$ with the prime ideals $\mathfrak{p}_i = \mathrm{rad}(\mathfrak{q}_i)$. Then $\mathcal{Z}(\mathfrak{q}_i)$ are affine algebraic varieties.

5.2 Maximal ideals in polynomial rings.

Lemma 5.2. *Let* A *be a commutative ring, let* $f \in A[x_1, ..., x_n]$. *If* $f(a_1, ..., a_n) = 0$ *for some* $(a_1, ..., a_n) \in k^n$, *then there exist polynomials* $g_1, ..., g_n \in A[x_1, ..., x_n]$ *such that*

$$f = (x_1 - a_1)g_1 + ... + (x_n - a_n)g_n.$$

Proposition 5.3. *Let k be algebraically closed and let $\mathfrak{m} \lhd k[x_1, \ldots, x_n]$. Then $\mathfrak{m}$ is maximal if and only if $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$, for some $a_1, \ldots, a_n \in k$.*

Corollary 5.4. *Let k be algebraically closed.*

1. *If $\mathfrak{m} \lhd k[x_1, ..., x_n]$ is a maximal ideal, then $\mathcal{Z}(\mathfrak{m})$ is a singleton.*
2. *For every $\underline{a} \in k^n$ the ideal $\mathcal{I}(\underline{a})$ is maximal.*

Proposition 5.5. *Let k be algebraically closed. The map*

$$\mathcal{I}\colon \operatorname{Var} k^n \to \operatorname{Spec} k[x_1, \ldots, x_n], \qquad V \mapsto \mathcal{I}(V)$$

is a bijection. In particular singleton sets are mapped onto maximal ideals, and the inverse map is given by

$$\mathcal{Z}\colon \operatorname{Spec} k[x_1, \ldots, x_n] \to \operatorname{Var} k^n, \qquad \mathfrak{p} \mapsto \mathcal{Z}(\mathfrak{p}).$$

5.3 Radical ideals.

Definition 5.6. *An ideal $\mathfrak{a}$ of a ring A is called **radical**, if $\mathfrak{a} = \mathrm{rad}(\mathfrak{a})$.*

Lemma 5.7. *Let* A *be a ring. An ideal* $\mathfrak{a} \lhd A$ *is radical if and only if the ring* $A/\mathfrak{a}$ *does not have nonzero nilpotents.*

Proposition 5.8. *Let k be algebraically closed and let $\mathfrak{a} \lhd k[x_1, \ldots, x_n]$. Then $\mathfrak{a}$ is radical if and only if $\mathfrak{a} = \mathcal{I}(V)$ for some affine algebraic set $V \subset k^n$.*

Proposition 5.9. *Let k be algebraically closed and let $\mathfrak{a} \lhd k[x_1, \ldots, x_n]$ be a radical ideal. Then $\mathfrak{a}$ has a unique decomposition into prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r \lhd k[x_1, \ldots, x_n]$:*

$$\mathfrak{a} = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r \quad with\, \mathfrak{p}_i \not\subseteq \mathfrak{p}_j,\, for\; i \neq j.$$