4 Affine algebraic varietes. Hilbert Nullstellensatz.

4.1 Affine algebraic varietes.

Definition 4.1. *A nonempty affine algebraic set* $V \subseteq k^n$ *will be called an* ***affine algebraic variety*** *if the ideal* $\mathcal{I}(V)$ *of the ring* $k[x_1, ..., x_n]$ *is prime.*

Definition 4.2. *A nonempty affine algebraic set* $V \subseteq k^n$ *will be called* ***irreducible****, if for affine algebraic sets* $A, B \subseteq k^n$*:*

$$V = A \cup B \Rightarrow V = A \vee V = B.$$

Theorem 4.3. *A nonempty affine algebraic set $V \subseteq k^n$ is irreducible if and only if it is an affine algebraic variety.*

Theorem 4.4. *Every affine algebraic set A is a finite sum of affine algebraic varieties:*

$$A = V_1 \cup \ldots \cup V_r, \quad r \geq 1.$$

If in the above decomposition the varieties V_i are incomparable (that is $V_i \not\subset V_j$ for $i \neq j$), then they are uniquely defined.

Remark 4.5. Let $V \subseteq k^n$ be an affine algebraic variety endowed with the Zariski topology inherited from k^n. Then in V every two nonempty open sets have a nonempty intersection.

Remark 4.6. Let $\operatorname{Spec} k[x_1, \ldots, x_n]$ denote the **prime spectrum** of the ring $k[x_1, \ldots, x_n]$, that is the set of all prime ideals. Let $\operatorname{Var} k^n$ denote the set of all affine algebraic varieties in k^n. The map

$$\mathcal{I}\colon \operatorname{Var} k^n \to \operatorname{Spec} k[x_1, \ldots, x_n], \quad V \mapsto \mathcal{I}(V)$$

is

1. injective,
2. surjective if and only if for every prime ideal $\mathfrak{p}$ of the ring $k[x_1, \ldots, x_n]$

$$\mathfrak{p} = \mathcal{I}(\mathcal{Z}(\mathfrak{p})).$$

4.2 Hilbert Nullstellensatz.

Remark 4.7. Let $\mathfrak{a} \lhd k[x_1, ..., x_n]$. Then

$$\mathfrak{a} \subseteq \operatorname{rad}(\mathfrak{a}) \subseteq \mathcal{I}(\mathcal{Z}(\mathfrak{a})).$$

Theorem 4.8. (Hilbert Nullstellensatz) *Let* k *be algebraically closed, let* $\mathfrak{a} \lhd k[x_1, \ldots, x_n]$*. Then* $\operatorname{rad}(\mathfrak{a}) = \mathcal{I}(\mathcal{Z}(\mathfrak{a}))$*.*

Definition 4.9. *Let B be a domain and let A be a subring of B. An element $x \in B$ is* ***integral*** *over A if there exist $a_1, \ldots, a_n \in A$ such that*

$$a_1 + a_2 x + \ldots + a_n x^{n-1} + x^n = 0;$$

The set of all elements of B integral over A will be denoted by $C_B(A)$ and called the ***integral closure*** *of A in B.*

Proposition 4.10. *Let B be a domain and let A be a subring of B. Then the integral closure $C_B(A)$ of A in B forms a ring.*

Definition 4.11. *Let A be a domain and B a subring of A. We say that A is **integrally closed in B** if $C_B(A) = A$. We say that A is **integrally closed**, if it is integrally closed in its own field of fractions.*

Remark 4.12. Let A be an UFD. Then A is integrally closed.

Lemma 4.13. *Let k be a subfield of a commutative ring with identity A and let $L = k[x_1, \ldots, x_n]$ be a subring of A generated by the elements $x_1, \ldots, x_n \in A$ over k. If L is a field, then L is a finite extension of k.*

Lemma 4.14. *Let k be algebraically closed, let $\mathfrak{a} \lhd k[x_1, \ldots, x_n]$ be a proper ideal. Then $\mathcal{Z}(\mathfrak{a}) \neq \emptyset$.*

Lemma 4.15. *Let k be algebraically closed, let $\mathfrak{a} = \langle f_1, \ldots, f_r \rangle \lhd k[x_1, \ldots, x_n]$. Then $\mathcal{Z}(\mathfrak{a}) = \emptyset$ if and only if there exist polynomials $h_1, \ldots, h_r \in k[x_1, \ldots, x_n]$ such that*

$$f_1 h_1 + \ldots + f_r h_r = 1.$$

Lemma 4.16. *Let k be algebraically closed, let $\mathfrak{a} \lhd k[x_1, \dots, x_n]$ and let $f \in \mathcal{I}(\mathcal{Z}(\mathfrak{a}))$. Then $f \in \mathrm{rad}(\mathfrak{a})$.*