3 Affine algebraic sets.

3.1 Affine algebraic sets and their ideals.

Let k be any field.

Definition 3.1. *A* ***zero*** *of a polynomial* $f \in k[x_1, \ldots, x_n]$ *in the affine space* k^n *is a point* $(a_1, \ldots, a_n) \in k^n$ *such that* $f(a_1, \ldots, a_n) = 0$.

An ***affine algebraic set*** V *is a subset of the affine space* k^n *consisting of all common zeros of some set of polynomials* $\mathcal{S} \subseteq k[x_1, \ldots, x_n]$:

$$V = \{(a_1, \ldots, a_n) \in k^n | \, f(a_1, \ldots, a_n) = 0 \ \text{ for all } f \in \mathcal{S}\}.$$

We shall call the set V *to be defined by the set of polynomials* $\mathcal{S}$ *and denote by* $V = \mathcal{Z}(\mathcal{S})$.

Remark 3.2. Let $\mathcal{S} \subseteq k[x_1, ..., x_n]$ and let $\mathfrak{a}$ be the ideal of $k[x_1, ..., x_n]$ generated by $\mathcal{S}$. Then

$$\mathcal{Z}(\mathcal{S}) = \mathcal{Z}(\mathfrak{a}).$$

Remark 3.3. Let $\mathcal{S} \subseteq k[x_1, \ldots, x_n]$. Then there exists a finite set $\{f_1, \ldots, f_r\} \subseteq k[x_1, \ldots, x_n]$ such that

$$\mathcal{Z}(\mathcal{S}) = \mathcal{Z}(f_1, \ldots, f_r).$$

Remark 3.4. Let $V \subseteq k^n$ be an affine algebraic set. The set $\mathcal{I}(V)$ of all polynomials whose common zeros coincide with V:

$$\mathcal{I}(V) = \{ f \in k[x_1, ..., x_n] | \, f(a_1, ..., a_n) = 0 \text{ for all } (a_1, ..., a_n) \in V \}$$

is an ideal of $k[x_1, ..., x_n]$.

Definition 3.5. *Let $V \subseteq k^n$ be an affine algebraic set. The ideal $\mathcal{I}(V)$ consisting of polynomials whose common zeros constitute V shall be called the* ***ideal of the affine algebraic set*** *V.*

Remark 3.6. Let V, V_1, $V_2 \subset k^n$ be affine algebraic sets in k^n, let $\mathfrak{a}$, $\mathfrak{a}_1$, $\mathfrak{a}_2$ be ideals of $k[x_1, ..., x_n]$. Then:

1. $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \Rightarrow \mathcal{Z}(\mathfrak{a}_1) \supseteq \mathcal{Z}(\mathfrak{a}_2)$,
2. $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) \supseteq \mathfrak{a}$,
3. $\mathcal{Z}(\mathcal{I}(V)) = V$,
4. $V_1 \subseteq V_2 \Leftrightarrow \mathcal{I}(V_1) \supseteq \mathcal{I}(V_2)$,
5. $V_1 = V_2 \Leftrightarrow \mathcal{I}(V_1) = \mathcal{I}(V_2)$.

Lemma 3.7. *Let $f, g \in k[x_1, x_2]$ and assume that f is irreducible in $k[x_1, x_2]$ and that $f \nmid g$. Then the system of equations*

$$f(x_1, x_2) = 0 \quad \textit{and} \quad g(x_1, x_2) = 0$$

has only a finite number of solutions in the field k.

Theorem 3.8. *Let* $f \in k[x_1, x_2]$ *be an irreducible polynomial in* $k[x_1, x_2]$*. If the curve* $\mathcal{Z}(f)$ *contains infinitely many points, then*

$$\mathcal{I}(\mathcal{Z}(f)) = \langle f \rangle.$$

3.2 Zariski topology.

Lemma 3.9. *A finite sum of affine algebraic sets is an affine algebraic set. To be more precise, let* $\mathfrak{a}_1, ..., \mathfrak{a}_m$ *be ideals of the ring* $k[x_1, ..., x_n]$*. Then*

$$\mathcal{Z}(\mathfrak{a}_1) \cup ... \cup \mathcal{Z}(\mathfrak{a}_m) = \mathcal{Z}(\mathfrak{a}_1 \cdot ... \cdot \mathfrak{a}_m),$$

where $\mathfrak{a}_1 \cdot ... \cdot \mathfrak{a}_m = \{\sum_{i=1}^{k} a_{i1}a_{i2}...a_{im} | \, k \in \mathbb{N}, a_{ij} \in \mathfrak{a}_j, j \in \{1, ..., m\}, i \in \{1, ..., k\}\}$.

Remark 3.10. Let $\mathfrak{a}_1, \dots, \mathfrak{a}_m$ be ideals of the ring $k[x_1, \dots, x_n]$. Then

$$\mathcal{Z}(\mathfrak{a}_1 \cdot \ldots \cdot \mathfrak{a}_m) = \mathcal{Z}(\mathfrak{a}_1 \cap \ldots \cap \mathfrak{a}_m).$$

Lemma 3.11. *Intersection of any number of affine algebraic sets if an affine algebraic set. To be more precise, let* $\{\mathfrak{a}_i |\, i \in I\}$ *be a family of ideals of the ring* $k[x_1, \ldots, x_n]$. *Then*

$$\bigcap_{i \in I} \mathcal{Z}(\mathfrak{a}_i) = \mathcal{Z}\left(\left\langle \bigcup_{i \in I} \mathfrak{a}_i \right\rangle\right).$$

Remark 3.12. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ be ideals of the ring $k[x_1, \ldots, x_n]$. Then

$$\mathcal{Z}(\mathfrak{a}_1 + \ldots + \mathfrak{a}_m) = \mathcal{Z}(\langle \mathfrak{a}_1 \cup \ldots \cup \mathfrak{a}_m \rangle).$$

Theorem 3.13. *In k^n there is a topology whose closed sets are affine algebraic sets in k^n.*

Definition 3.14. *The topology of k^n defined by affine algebraic sets is called the* ***Zariski topology*** *in k^n.*

Remark 3.15. In every nonempty family of affine algebraic sets there exists a minimal affine algebraic set.

Remark 3.16. Every affine algebraic set $V \subseteq k^n$ is compact in the Zariski topology.