2 Primary decomposition.

2.1 Primary decomposition.

Remark 2.1. Consider the ring $\mathbb{Z}$ and an element $n \in \mathbb{Z}$. Then there exist uniquely determined prime numbers $p_1, \ldots, p_m$ and exponents $k_1, \ldots, k_m \in \mathbb{N}$ such that

$$n = \pm p_1^{k_1} \cdot \ldots \cdot p_m^{k_m}$$

or, equivalently:

$$(n) = (p_1^{k_1}) \cdot \ldots \cdot (p_m^{k_m}) = (p_1^{k_1}) \cap \ldots \cap (p_m^{k_m}).$$

Definition 2.2. *Let R be any ring. An ideal $\mathfrak{q} \lhd R$ is called* ***primary****, if $\mathfrak{q} \neq R$ and for all $a, b \in R$*

$$ab \in \mathfrak{q} \land b \notin \mathfrak{q} \Rightarrow \exists n \in \mathbb{N} \quad a^n \in \mathfrak{q}.$$

Example 2.3.

1. Every prime ideal is primary.
2. An ideal in $\mathbb{Z}$ generated by a power of a prime number is primary.
3. Let R be a principal ideal domain. Then $\mathfrak{q}$ is primary if and only if $\mathfrak{q} = \mathfrak{p}^n$, for a prime ideal $\mathfrak{p}$.

Lemma 2.4. *Let R be a ring, let $\mathfrak{q} \lhd R$ be a proper ideal in R. The following conditions are equivalent:*

i. *$\mathfrak{q}$ is primary,*

ii. *every zero divisor in $R/\mathfrak{q}$ is nilpotent,*

iii. *the zero ideal in $R/\mathfrak{q}$ is primary.*

Example 2.5. The ideal $(x, y^2) \lhd k[x, y]$, where k is any field, is primary, but is not a power of a prime ideal.

Definition 2.6. *Let R be a ring. An ideal $\mathfrak{n} \lhd R$, $0 \neq \mathfrak{n}$ is* ***irreducible*** *if, for all $\mathfrak{a}, \mathfrak{b} \lhd R$*

$$\mathfrak{n} = \mathfrak{a} \cap \mathfrak{b} \Rightarrow \mathfrak{n} = \mathfrak{a} \vee \mathfrak{n} = \mathfrak{b}.$$

Example 2.7.

1. Every maximal ideal is irreducible.
2. Every prime ideal is irreducible.
3. An ideal $\mathfrak{n} \lhd R$ is irreducible if and only if the zero ideal in $R/\mathfrak{n}$ is irreducible.

Lemma 2.8. *Let R be Noetherian. Every irreducible ideal in R is primary.*

Example 2.9. The ideal $(4, 2x, x^2) \lhd \mathbb{Z}[x]$ is primary, but not irreducible.

Lemma 2.10. *Let R be Noetherian, let $\mathfrak{a} \lhd R$ be a proper ideal. Then $\mathfrak{a}$ is an intersection of a finite number of irreducible ideals.*

Theorem 2.11. *Let R be Noetherian, let $\mathfrak{a} \lhd R$ be a proper ideal. Then $\mathfrak{a}$ is an intersection of a finite number of primary ideals.*

2.2 Radical of an ideal.

Definition 2.12. *Let R be a ring, let $\mathfrak{a} \lhd R$. The* ***radical*** *of the ideal $\mathfrak{a}$ is defined to be*

$$\operatorname{rad}\mathfrak{a} = \{r \in R \mid \exists n \in \mathbb{N}\, r^n \in \mathfrak{a}\}.$$

Remark 2.13. Let R be a ring, let $\mathfrak{a} \lhd R$. Then $\operatorname{rad} \mathfrak{a}$ is an ideal.

Remark 2.14. Let R be a ring, let $\mathfrak{a}, \mathfrak{b} \lhd R$.

1. $\mathfrak{a} \subseteq \operatorname{rad} \mathfrak{a}$,
2. $\mathfrak{a} \subseteq \mathfrak{b} \Rightarrow \operatorname{rad} \mathfrak{a} \subseteq \operatorname{rad} \mathfrak{b}$,
3. $\operatorname{rad} (\operatorname{rad} \mathfrak{a}) = \operatorname{rad} \mathfrak{a}$,
4. $\operatorname{rad} \mathfrak{a} \cdot \mathfrak{b} = \operatorname{rad} \mathfrak{a} \cap \mathfrak{b}$,
5. $\operatorname{rad} \mathfrak{a} \cap \mathfrak{b} = \operatorname{rad} \mathfrak{a} \cap \operatorname{rad} \mathfrak{b}$,
6. $\operatorname{rad} \mathfrak{a} = (1) \Leftrightarrow \mathfrak{a} = (1)$,
7. $\operatorname{rad} \mathfrak{a} + \mathfrak{b} = \operatorname{rad}(\operatorname{rad} \mathfrak{a} + \operatorname{rad} \mathfrak{b})$,
8. $\mathfrak{a} + \mathfrak{b} = (1) \Leftrightarrow \operatorname{rad} \mathfrak{a} + \operatorname{rad} \mathfrak{b} = (1)$.

Remark 2.15. Let R be a ring, let $\mathfrak{p} \lhd R$ be a prime ideal, let $m \in \mathbb{N}$. Then $\operatorname{rad} \mathfrak{p}^m = \mathfrak{p}$.

Remark 2.16. Let R be a ring, let $\mathfrak{a} \lhd R$. If $\operatorname{rad} \mathfrak{a}$ is a maximal ideal, then $\mathfrak{a}$ is primary.

Lemma 2.17. *Let R be a ring, let $\mathfrak{q} \lhd R$ be a primary ideal. Then $\operatorname{rad} \mathfrak{q}$ is prime.*

Definition 2.18. *Let R be a ring, let $\mathfrak{q} \lhd R$ be a primary ideal and let $\mathfrak{p} = \operatorname{rad} \mathfrak{q}$. Then $\mathfrak{q}$ is called $\mathfrak{p}$-**primary**.*

Remark 2.19. Let R be a ring, let $\mathfrak{m} \lhd R$ be a maximal ideal, let $m \in \mathbb{N}$. Then $\mathfrak{m}^m$ is $\mathfrak{m}$-primary.

Lemma 2.20. *Let R be a ring, let $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ be $\mathfrak{p}$-primary. Then $\mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$ is $\mathfrak{p}$-primary.*

Definition 2.21. *Let R be a ring, let $\mathfrak{a} \lhd R$ be a proper ideal and let*

$$\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$$

be a primary decomposition of $\mathfrak{a}$. If

$$\mathfrak{q}_j \not\supseteq \bigcap_{i \neq j} \mathfrak{q}_i$$

and

$$\operatorname{rad} \mathfrak{q}_i \neq \operatorname{rad} \mathfrak{q}_j \ \textit{for } i \neq j,$$

*then the primary decomposition $\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$ is called **minimal**.*

Theorem 2.22. (Noether-Lasker) *Let R be a Noetherian ring, let $\mathfrak{a} \lhd R$ be a proper ideal. Then $\mathfrak{q}$ has a minimal primary decomposition and the prime ideals $\mathfrak{p}_i = \operatorname{rad} \mathfrak{q}_i$ are uniquely determined up to the order.*