## 4 Affine algebraic varieties. Hilbert Nullstellensatz.

1. Let $\mathfrak{a} \lhd k[x_1, ..., x_n]$, where $k$ is algebraically closed. Show that $f \in \operatorname{rad} \mathfrak{a}$ if and only if $1 \in (\mathfrak{a} \cup \{1 - z \cdot f\}) \lhd k[x_1, ..., x_n, z]$.

2. Let $\mathfrak{a} = (x^2 + y^2 + z^2, xy + xz + yz) \lhd \mathbb{C}[x, y, z]$. Describe $\mathcal{Z}(\mathfrak{a})$ and show that $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) \neq \mathfrak{a}$.

3. Let $f, g \in k[x, y]$, where $k$ is algebraically closed. Show that $\mathcal{Z}(f) = \mathcal{Z}(g)$ if and only if there exist integers $m, n \in \mathbb{N}$ such that $f | g^n$ and $g | f^m$.

4. Let $f \in k[x, y]$, where $k$ is algebraically closed and let $f = f_1^{k_1} \cdot ... \cdot f_m^{k_m}$ be a factorization of $f$ into irreducible polynomials with coefficients in $k$. Show that

   a) $\mathcal{Z}(f) = \mathcal{Z}(f_1) \cup ... \cup \mathcal{Z}(f_m)$,

   b) $\mathcal{I}(\mathcal{Z}(f)) = (f_1 \cdot ... \cdot f_r)$.

5. Show that every proper radical ideal, i.e. such that $\operatorname{rad}(\mathfrak{a}) = \mathfrak{a}$, is the intersection of all prime ideals that contain it.

**Homework:** Problems 3, 4 and 5.