3 Affine algebraic sets. Zariski topology.

1. Show an example of an ideal $\mathfrak{a} \lhd k[x_1, ..., x_n]$ such that $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) \supsetneq \mathfrak{a}$.

2. Find $\mathcal{I}(\mathcal{Z}(f))$, where $f(x, y) = xy - 1 \in \mathbb{R}[x, y]$.

3. Let $A \subseteq k^n$ and $B \subseteq k^m$ be affine algebraic sets. Show that $A \times B \subseteq k^{n+m}$ is an affine algebraic set.

4. Let $A_1, ..., A_m \subseteq k^n$ be algebraic sets. Show that
$$\mathcal{I}(A_1 \cup ... \cup A_m) = \bigcap_{i=1}^{m} \mathcal{I}(A_i).$$

5. Show that the space k^n equipped with a Zariski topology is T_1.

6. Show that every set $A \subseteq \mathbb{C}^n$ closed in the Zariski topology is also closed in the natural topology of $\mathbb{C}^n$.

7. Show an example of a set $A \subseteq \mathbb{R}^n$ closed in the natural topology of $\mathbb{R}^n$ which is not closed in the Zariski topology.

Homework: Problem 5, 6 and 7.