2 Primary decomposition.

1. Show that the ideal $(x^2, y^2) \lhd k[x, y]$, where k is any field, is not a product of prime ideals.

2. Show that in a domain the zero ideal is irreducible.

3. Show that the ideal $(4, 2x, x^2) \lhd \mathbb{Z}[x]$ is primary, but not irreducible.

4. Show that the ideal $\mathfrak{q} = (4, x) \lhd \mathbb{Z}[x]$ is primary, but is not a power of a prime ideal.

5. Let $\mathfrak{q} \lhd R$ be a primary ideal. Show that $\operatorname{rad} \mathfrak{q}$ is the smallest prime ideal of R containing the ideal $\mathfrak{q}$.

6. Let $R = \{a_0 + a_1 x + \ldots + a_n x^n \in \mathbb{Z}[x] \mid a_1 \equiv 0 \pmod 3\}$.

    a) Show that in the ring R the ideal $\mathfrak{p} = (3x, x^2, x^3)$ is prime.

    b) Show that the ideal $\mathfrak{p}^2$ is not primary.

7. Let $\mathfrak{q} = (2, x)^2 \lhd \mathbb{Z}[x]$.

    a) Show $\mathfrak{q}$ is primary.

    b) Show that $\mathfrak{q} = (4, x) \cap (2, x^2)$.

Homework: Problems 5, 6 and 7.