1 Noetherian rings.

1. Show that if I is a nonzero ideal in a principal ideal domain R, then R/I is Noetherian.

2. Let S be a multiplicative subset of a commutative Noetherian ring with identity. Show that the ring $S^{-1}R$ is Noetherian.

3. Show that a ring R is Noetherian if and only if the ring $\mathrm{Mat}_n R$ is Noetherian for every $n \geqslant 1$.

4. Let R be a Noetherian ring and let φ: $R \to R$ be an epimorphism. Show that φ is an isomorphism.

5. Let R be a Noetherian ring, let $\mathfrak{a} \lhd R$. The **radical** of the ideal $\mathfrak{a}$ is defined to be

$$\mathrm{rad}\,\mathfrak{a} = \{r \in R | \, \exists n \in \mathbb{N}\, r^n \in \mathfrak{a}\}.$$

   Show that there is an integer $m \in \mathbb{N}$ such that $(\mathrm{rad}\,\mathfrak{a})^m \subseteq \mathfrak{a}$.

6. Let R be a Noetherian ring. The **nilradical** of R is defined to be

$$\mathrm{Nil}\, R = \{r \in R | \, \exists n \in \mathbb{N}\, r^n = 0\}.$$

   Show that there is an integer $m \in \mathbb{N}$ such that $(\mathrm{Nil}\, R)^m = (0)$.

7. Let R be a Noetherian ring, let $\mathfrak{p} \lhd R$ be a prime ideal. An ideal $\mathfrak{q} \lhd R$ is called **$\mathfrak{p}$-primary** if $\mathfrak{p} = \mathrm{rad}\,\mathfrak{q}$ and for all $r, s \in R$

$$r s \in \mathfrak{q} \wedge s \notin \mathfrak{q} \Rightarrow \exists n \in \mathbb{N}\, r^n \in \mathfrak{q}.$$

   Show that if $\mathfrak{q}$ is $\mathfrak{p}$-primary, then there is an integer $m \in \mathbb{N}$ such that $\mathfrak{p}^m \subseteq \mathfrak{q} \subseteq \mathfrak{p}$.

Homework: Problems 5, 6 and 7.