7 Morphisms of affine algebraic sets. Category of affine algebraic sets. Isomorphisms of affine algebraic sets.

7.1 Morphisms of affine algebraic sets.

Definition 7.1. *Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine algebraic sets. A* ***morphism*** *$f: V \to W$ is a map such that there exist $f_1, \ldots, f_m \in k[V]$ such that $f(a) = (f_1(a), \ldots, f_m(a))$, for all $a \in V$.*

Remark 7.2. Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine algebraic sets, let $f_1, \ldots, f_m \in k[V]$. Then $f = (f_1, \ldots, f_m): V \to W$ is a morphism if and only if

$$g(f_1, \ldots, f_m) = 0 \in k[V] \quad \text{for all } g \in \mathcal{I}(W).$$

Proof. Indeed, one easily checks that

$$\begin{aligned}
(f_1(a_1, \ldots, a_n), \ldots, f_m(a_1, \ldots, a_n)) \in W &\Leftrightarrow g(f_1(a_1, \ldots, a_n), \ldots, f_m(a_1, \ldots, a_n)) = 0 \text{ for all } g \in \mathcal{I}(W) \\
&\Leftrightarrow g(f_1, \ldots, f_m)(a_1, \ldots, a_n) = 0 \text{ for all } g \in \mathcal{I}(W) \\
&\Leftrightarrow g(f_1, \ldots, f_m) \in \mathcal{I}(V) \text{ for all } g \in \mathcal{I}(W) \\
&\Leftrightarrow g(f_1, \ldots, f_m) = 0 \in k[V] \text{ for all } g \in \mathcal{I}(W).
\end{aligned}$$

□

Example 7.3. Consider the following easy examples.

- Let $f \in k[V]$. Then $f: V \to k$ is a morphism.
- Let $f: k^n \to k^m$ be a linear map. Then f is a morphism.
- Let $f: \mathcal{Z}(xy - 1) \to k$ be given by $f(x, y) = x$. Then f is a morphism.
- Let $f: k \to \mathcal{Z}(y^2 - x^3)$ be given by $f(t) = (t^2, t^3)$. Then f is a morphism.

7.2 Category of affine algebraic sets.

Definition 7.4. *A* ***category*** *$\mathcal{C}$ consists of a class of* ***objects*** *$\mathrm{Ob}(\mathcal{C})$, denoted by $A, B, C, \ldots$ and a class of* ***morphisms*** *(or* ***arrows****) $\mathrm{Ar}(\mathcal{C})$ together with:*

1. *classes of pairwise disjoint arrows $\mathrm{Hom}(A, B)$, one for each pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$; and elements f of the class $\mathrm{Hom}(A, B)$ shall be called a* ***morphism*** *from A to B and denoted by $A \xrightarrow{f} B$ or $f: A \to B$,*
2. *functions $\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$, for each triple of objects $A, B, C \in \mathrm{Ob}(\mathcal{C})$, called* ***composition*** *of morphisms; for morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ values of this function shall be denoted by $(g, f) \mapsto g \circ f$, and the morphism $A \xrightarrow{g \circ f} C$ shall be called the composition of morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$.*

Moreover, we require that the following two axioms hold true:

Associativity. *If $A \xrightarrow{f} B$, $B \xrightarrow{g} C$ and $C \xrightarrow{h} D$ are morphisms in $\mathcal{C}$, then*

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Identity. *For every object $B \in \mathrm{Ob}(\mathcal{C})$ there exists a morphism $B \xrightarrow{1_B} B$ such that for all morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$*

$$1_B \circ f = f \text{ and } g \circ 1_B = g.$$

*If the classes $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Ar}(\mathcal{C})$ are sets, we shall call the category $\mathcal{C}$ **small**. If all classes $\mathrm{Hom}(A, B)$ are sets, we shall call the category $\mathcal{C}$ **locally small.***

Example 7.5.

1. We shall set the notation for a number of familiar categories here:
   - $\mathcal{S}$et is the category of sets with functions as morphisms;
   - $\mathcal{G}$rp is the category of groups with group homomorphisms as morphisms;
   - $\mathcal{T}$op is the category of topological spaces with continuous functions as morphisms.
   - $\mathcal{A}b$ is the category of Abelian groups;
   - $\mathcal{R}$ng is the category of rigns;
   - $\mathcal{F}$ield is the category of fields;
   - $k-\mathcal{V}$ect is the category of $k-$vector spaces.

   All these categories are locally small, but not small.

2. The notion of a category allows for a different take on familiar constructions in mathematics. For example, consider a partial order $(P, \leqslant)$. One checks that considering the elements of P as objects, and defining morphisms by

   $$a \to b \qquad \Leftrightarrow \qquad a \leqslant b$$

   one obains a category, which is small provided P is a set.

3. Affine algebraic sets with morphisms defined in the previous section for a category.

7.3 Isomorphisms of affine algebraic sets.

Definition 7.6. *Let $V \subseteq k^n$ and $W \subseteq k^m$ be affine algebraic sets. A morphism $f: V \to W$ shall be called an **isomorphism** is there exists a morphism $g: W \to V$ such that*

$$f \circ g = \mathrm{id}_W \quad \text{and} \quad g \circ f = \mathrm{id}_V.$$

*If there exists an isomorphism $f: V \to W$, we shall call the sets V and W **isomorphic** and denote $V \cong W$.*

Example 7.7. One easily checks that:

- $\mathcal{Z}(y - x^k) \cong k$ via $f(x, y) = x$ and $g(t) = (t, t^k)$;
- $f: \mathcal{Z}(xy - 1) \to k$ given by $f(x, y) = x$ is not an isomorphism;
- $f: k \to \mathcal{Z}(y^2 - x^3)$ given by $f(t) = (t^2, t^3)$ is not an isomorphism, even though it is a bijection.