6 Coordinate ring of an affine algebraic set.

6.1 Coordinate ring of an affine algebraic set.

Definition 6.1. *Let k be a field, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V. The ring $k[V] := k[x_1, ..., x_n] / \mathcal{I}(V)$ is called the **coordinate ring** of V.*

Remark 6.2. Let k be a field, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V. Let $f \in k[x_1, ..., x_n]$. The polynomial f defines a polynomial function $k^n \to k$. Let f_V be the restriction of f to the set V, $f_V = f\restriction_V$. Then $f_V = g_V$ if and only if $f + \mathcal{I}(V) = g + \mathcal{I}(V)$.

Proof. Indeed, $f_V = g_V$ means that $f(a_1, ..., a_n) = g(a_1, ..., a_n)$, for all $(a_1, ..., a_n) \in V$, that is $(f - g)(a_1, ..., a_n) = 0$, for all $(a_1, ..., a_n) \in V$, or, equivalently, $f - g \in \mathcal{I}(V)$. □

Remark 6.3. Let k be a field, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V. Let κ: $k[x_1, ..., x_n] \to k[V]$ be the canonical epimorphism, $\kappa(f) = \overline{f} := f + \mathcal{I}(V)$. Then $k[V]$ is a k-ring finitely generated over k by $\overline{x_1}, ..., \overline{x_2}$.

Remark 6.4. Let k be algebraically closed, $V \subseteq k^n$ an affine algebraic set, $\mathcal{I}(V)$ the ideal of V. Then $k[V]$ has no nonzero nilpotents.

Proof. By Hilbert Nullstellensatz, $\mathcal{I}(V)$ is radical, so that, by Lemma 5.7, $k[V] = k[x_1, ..., x_n] / \mathcal{I}(V)$ has no nonzero nilpotents. □

Theorem 6.5. *Let k be algebraically closed. Then a k-ring A is isomorphic to a coordinate ring of an affine algebraic set $V \subseteq k^n$ if and only if it is finitely generated over k and has no nonzero nilpotents.*

Proof. Let $A = k[t_1, ..., t_n]$ be a ring finitely generated over k with no nonzero nilpotents. The map

$$k[x_1, ..., x_n] \to A, \qquad f \mapsto f(t_1, ..., t_n)$$

is a well-defined ring epimorphism. Define by $\mathfrak{a}$ its kernel. The ring $k[x_1, ..., x_n] / \mathfrak{a} \cong A$ has no nonzero nilpotents, hence, by Lemma 5.7, the ideal $\mathfrak{a}$ is radical. Thus $\mathfrak{a} = \mathcal{I}(\mathcal{Z}(\mathfrak{a}))$ and, consequently, $A \cong k[\mathcal{Z}(\mathfrak{a})]$. □

Example 6.6. One easily checks that:

- $V = k^n$, $k[V] \cong k[x_1, ..., x_n]$;
- $V = \emptyset$, $k[V] \cong 0$;
- $V = \{(a_1, ..., a_n)\}$, $k[V] \cong k$.

Example 6.7. Let $V = \mathcal{Z}(f)$, where $f \in k[x_1, ..., x_n]$ is square-free and k is algebraically closed. Then $k[V] \cong k[x_1, ..., x_n] / (f) \cong k[\alpha_1, ..., \alpha_n]$ where $f(\alpha_1, ..., \alpha_n) = 0$.

Proof. By Hilbert Nullstellensatz $\mathcal{I}(V) = \mathcal{I}(\mathcal{Z}(f)) = \mathrm{rad}\,(f)$. One easily checks that $(f) = \mathrm{rad}\,(f)$ if and only if f is square-free, which follows that $k[V] \cong k[x_1, ..., x_n] / (f) \cong k[\alpha_1, ..., \alpha_n]$, where $\alpha_i = x_i + (f)$, for $i \in \{1, ..., n\}$. □

Example 6.8. Let $V = \mathcal{Z}(a_1x_1 + \ldots + a_nx_n - b)$, where $a_1, \ldots, a_n, b \in k$ and k is algebraically closed. Then $k[V] \cong k[x_1, \ldots, x_{n-1}]$.

Proof. As in the previous example, $k[V] \cong k[x_1, \ldots, x_n] / (a_1x_1 + \ldots + a_nx_n - b) \cong k[\alpha_1, \ldots, \alpha_n]$, where $a_1\alpha_1 + \ldots + a_n\alpha_n = b$. Relabelling, if necessary, we may assume that $a_n \neq 0$. Further, we may assume that $a_n = 1$, since $\mathcal{Z}(a_1x_1 + \ldots + a_nx_n - b) = \mathcal{Z}\left(\frac{a_1}{a_n}x_1 + \ldots + \frac{a_n}{a_n}x_n - \frac{b}{a_n}\right)$. Thus

$$\alpha_n = b - a_1\alpha_1 - \ldots - a_{n-1}\alpha_{n-1},$$

and the ring $k[V] \cong k[\alpha_1, \ldots, \alpha_n]$ is generated by the elements $\alpha_1, \ldots, \alpha_{n-1}$. If suffices to show that these elements are algebraically independent: indeed, if $g(\alpha_1, \ldots, \alpha_{n-1}) = 0$, for some $g \in k[x_1, \ldots, x_{n-1}]$, then

$$\begin{aligned} \mathcal{I}(V) &= 0_{k[V]} = g(\alpha_1, \ldots, \alpha_{n-1}) = g(x_1 + \mathcal{I}(V), \ldots, x_{n-1} + \mathcal{I}(V)) \\ &= g(x_1, \ldots, x_{n-1}) + \mathcal{I}(V), \end{aligned}$$

so that $g \in \mathcal{I}(V) = (h)$, that is h divides g in the ring $k[x_1, \ldots, x_n]$. But this is impossible, since x_n appears in h with a nonzero coefficient, and does not appear in any of the monomials of g. Thus $\alpha_1, \ldots, \alpha_{n-1}$ are algebraically independent over k, and thus $k[\alpha_1, \ldots, \alpha_{n-1}] \cong k[x_1, \ldots, x_{n-1}]$. $\square$