1 Noetherian rings.

1.1 Noetherian rings.

Theorem 1.1. *Let R be a ring. The following conditions are quivalent:*

(FG). [1.1] *Every ideal of the ring R is finitely generated.*

(ACC). [1.2] *Every ascending chain of ideals in R is finite.*

(MAX). *Every nonempty family of ideals of the ring R has a maximal element.*

Proof. $\boldsymbol{(FG)} \Rightarrow \boldsymbol{(ACC)}$**:** Let $I_1 \subsetneq I_2 \subsetneq \ldots$ be an ascending chain of ideals in R. Let $J = \bigcup_{i=1}^{\infty} I_i$. Then $J \lhd R$ and, by $\boldsymbol{(FG)}$, $J = (a_1, \ldots, a_n)$. Thus for every $k \in \{1, \ldots, n\}$ there exists I_{i_k} such that $a_k \in I_{i_k}$. Let $m = \max \{i_k | \, k \in \{1, \ldots, n\}\}$. Then $a_1, \ldots, a_n \in I_m$, so that $J = (a_1, \ldots, a_n) \subset I_m$. On the other hand $I_m \subset \bigcup_{i=1}^{\infty} I_i = J$, and hence $J = I_m$. Moreover

$$J = I_m \subset I_{m+1} \subset I_{m+2} \subset \ldots \subset \bigcup_{i=1}^{\infty} I_i = J,$$

so $I_l = I_m = J$, for $l > m$.

$\boldsymbol{(ACC)} \Rightarrow \boldsymbol{(MAX)}$**:** Let $\mathcal{R} \neq \emptyset$ be a nonempty family of ideals of the ring R. Fix $I_1 \in R$. If I_1 is not maximal in $\mathcal{R}$, then there is $I_2 \in \mathcal{R}$ such that $I_1 \subsetneq I_2$. If I_2 is not maximal in $\mathcal{R}$, then there is $I_3 \in \mathcal{R}$ such that $I_2 \subsetneq I_3$. Continuing that way, had we not came across an ideal maximal in $\mathcal{R}$, we would eventually build an infitite ascending chain of ideals $I_1 \subsetneq I_2 \subsetneq \ldots$, contrary to $\boldsymbol{(ACC)}$.

$\boldsymbol{(MAX)} \Rightarrow \boldsymbol{(FG)}$**:** Let $J \lhd R$. Fix $a_1 \in J$. If $(a_1) \neq J$, then there is an $a_2 \in J \setminus (a_1)$. If $(a_1, a_2) \neq J$, then there is an $a_3 \in J \setminus \{a_1, a_2\}$. Continuing that way, we eventually obtain a family $\mathcal{R} = \{(a_1, \ldots, a_t) | \, (a_1, \ldots, a_t) \subset J, t \in \mathbb{N}\}$. By $\boldsymbol{(MAX)}$, $\mathcal{R}$ contains a maximal element $(a_1, \ldots, a_r)$, so that every element $a \in J$ belongs to $(a_1, \ldots, a_r)$. Hence $J = (a_1, \ldots, a_r)$. ☐

Definition 1.2. *Let R be a ring. If one (and hence every) condition of Theorem 1.1 is satisfied, then R is called a* ***noetherian ring.***

Lemma 1.3. *Let R and S be rings and let R be noetherian. Let $\varphi: R \to S$ be an epimorphism. Then S is noetherian.*

Proof. Let $J \lhd S$. Then $\varphi^{-1}(J) \lhd R$ is finitely generated, $\varphi^{-1}(J) = (a_1, \ldots, a_n)$. As φ is an epimorphism, $J = \varphi \circ \varphi^{-1}(J) = \varphi((a_1, \ldots, a_n)) = (\varphi(a_1), \ldots, \varphi(a_n))$ is finitely generated. ☐

Corollary 1.4. *Let R be noetherian, let $I \lhd R$. Then R/I is noetherian.*

Proof. R/I is a surjective image of R via the canonical epimorphism $\kappa: R \to R/I$. ☐

1.2 Hilbert basis theorem.

Theorem 1.5. (Hilbert basis theorem) *Let R be noetherian. Then $R[x]$ is noetherian.*

1.1. finitely generated

1.2. ascending chain condition

Proof. We shall shwo that $R[x]$ satisfies ***(FG)***. For that purpose, fix an $I \lhd R[x]$. As I can be decomposed into the union of sets consisting of polynomials of fixed degrees, let

$$I_i = \{a \in R | \exists_{a_0, \ldots, a_{i-i} \in R} a x^i + a_{i-1} x^{i-1} + \ldots + a_1 x + a_0 \in I\} \cup \{0\}, i \in \mathbb{N}.$$

One easily checks that $I_i \lhd R$. Observe that $I_i \subseteq I_{i+1}$, for $i \in \mathbb{N}$. Indeed, fix an $i \in \mathbb{N}$. If $f = a x^i + a_{i-1} x^{i-1} + \ldots + a_1 x + a_0 \in I$ and $a \in I_i$, then $x f = a x^{i+1} + a_{i-1} x^i + \ldots + a_1 x^2 + a_0 x \in I$ and hence $a \in I_{i+1}$.

Since R is noetherian, by ***(ACC)*** there exists a $r \in \mathbb{N}$ such that $I_r = I_{r+1} = \ldots$. By ***(FG)***:

$$\begin{aligned} I_0 &= (a_{01}, \ldots, a_{0n}) \\ I_1 &= (a_{11}, \ldots, a_{1n}) \\ &\vdots \\ I_r &= (a_{r1}, \ldots, a_{rn}), \end{aligned}$$

where, for the sake of simplicity, we allow some of the a_{ij} to be 0. Let

$$f_{ij} = a_{ij} x^i + a_{i-1}^{(ij)} x^{i-1} + \ldots + a_1^{(ij)} x + a_0^{(ij)} \in I.$$

It suffices to show that $I = (f_{01}, \ldots, f_{0n}, f_{11}, \ldots, f_{1n}, \ldots, f_{r1}, \ldots, f_{rn})$. The inclusion $(\supset)$ is obvious, and for the other one denote $J = (f_{01}, \ldots, f_{0n}, f_{11}, \ldots, f_{1n}, \ldots, f_{r1}, \ldots, f_{rn})$. Fix $f \in I$ and let $\deg f = d$. We shall proceed by induction on d. If $d = 0$, then $f = a$, for some $a \in R$, so that $f \in (f_{01}, \ldots, f_{0n})$.

For $d \geqslant 1$, assume that for all polynomials $g \in I$ of degree less than d, $g \in J$. If $r \geqslant d$, then there are $e_1, \ldots, e_n \in R$ such that

$$h = f - (e_1 f_{d1} + \ldots + e_n f_{dn}) \in I$$

and $\deg h < d$. Therefore $h \in J$ and $f \in J$.

If $r < d$, then $\deg x^{d-r} f_{r1} = \ldots = \deg x^{d-r} f_{rn} = d$ and the ideal I_r is being generated by the leading coefficients of these polynomials. Since $I_r = I_d$ and the leading coefficient of f belongs to I_d, it is a linear combination of the generators of I_r. Thus there are $c_1, \ldots, c_n \in R$ such that

$$g = f - (c_1 x^{d-r} f_{r1} + \ldots + c_n x^{d-r} f_{rn}) \in I$$

and $\deg g < d$. Therefore $g \in J$ and $f \in J$. □

Corollary 1.6. *Let R be noetherian. Then $R[x_1, \ldots, x_n]$ is noetherian.*