Problem set 20: double integrals.

(1) Estimate the volume of the solid that lies above the square $R = [0,2] \times [0,2]$ and below the paraboloid $z = 64 - 2x^2 - 2y^2$. Divide R into four equal squares and choose the sample points to be the upper right corners of each subsquare.

(2) Let $R = \{(x,y) : -1 \le x \le 1, -1 \le y \le 1\}$. Evaluate

$$\iint_R \sqrt{1-y^2}\, dA.$$

(3) Assume that $f(x,y) = C$, where C is some real constant. Suppose that $R = [a,b] \times [c,d]$ is a rectangle. Show that

$$\iint_R f(x,y)\, dA = C(b-a)(d-c).$$

(4) Evaluate the iterated integrals $\int_0^2 \int_0^1 x^2y^2\, dx\, dy$ and $\int_0^1 \int_0^2 x^2y^2\, dy\, dx$.

(5) Calculate $\iint_R x\sin(xy)\, dA$, where $R = [0,\pi] \times [1,2]$.

(6) Find the volume of the solid that lies under the plane $6x + 2y + z = 24$ and above the rectangle $R = \{(x,y) : 0 \le x \le 1, 0 \le y \le 20\}$.

(7) Evaluate $\iint_\Omega (2x+6y)\, dA$, where Ω is the region bounded by $y = 5x^2$ and $y = 4 + x^2$.

(8) Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region Ω bounded by the line $y = 4x$ and $y = x^2$.

(9) Determine $\iint_\Omega x^2y\, dA$ where Ω is the region bounded by $y = x - 2$ and $y^2 = x + 4$.

(10) Evaluate the iterated integral $\int_0^1 \int_x^1 y\, dy\, dx$.

(11) Evaluate the iterated integral $\int_0^{\pi/2} \int_0^{\sin\theta} e^{\cos\theta}\, dr\, d\theta$.

(12) Evaluate the integral $\iint_\Omega (3x+4y)\, dA$, where Ω is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 2$ and $x^2 + y^2 = 3$.

(13) Compute $\iint_{\mathbb{R}^2} e^{-x^2-y^2}\, dA$.

(14) Find the volume of the solid bounded by the plane $z = 2$ and the paraboloid $z = 3 - x^2 - y^2$.

(15) Let the curve $r = \cos(2\theta)$ be given in polar coordinates. The graph is a four-leaved rose. What is the area of all four leaves together?

(16) Suppose we are given a triangle Δ with corners $(0,0)$, $(1,0)$, and $(0,1)$. We further assume that the density is given by $\rho(x,y) = 1 + 2x + 4y$. What is the mass of the triangle?

(17) Assume that two random variables X and Y are given. Show that

$$f(x,y) = \begin{cases} 4xy, & \text{if } 0 \le x \le 1,\ 0 \le y \le 1, \\ 0, & \text{otherwise} \end{cases}$$

is a joint density. Then calculate $P(Y \le \frac{1}{2})$ and $P(X \le \frac{1}{4}, Y \ge \frac{1}{2})$.

(18) Assume we are given a lamina occupying the region Ω and having density function $\rho(x,y)$. If $(\overline{x}, \overline{y})$ are the coordinates of the center of mass, then denoting the mass

by m, we have

$$\overline{x} = \frac{1}{m} \iint_\Omega x\rho(x, y)\, dA,$$
$$\overline{y} = \frac{1}{m} \iint_\Omega y\rho(x, y)\, dA.$$

Assume that we are given a semicircular lamina whose radius is 1 and whose density is $\rho(x, y) = \sqrt{x^2 + y^2}$. What is the center of mass?

(19) The moment of inertia I_x of a lamina occupying the region Ω and having density function $\rho(x, y)$ about the x-axis is

$$I_x = \iint_\Omega y^2\rho(x, y)\, dA.$$

Similarly for the y-axis. The moment of inertia I_0 about the origin is

$$\iint_\Omega (x^2 + y^2)\rho(x, y)\, dA.$$

Suppose Ω is a disk centered around the origin with radius 1. Give the moments of inertia I_x, I_y, and I_0 if the density is constant.

(20) Let $f\colon (1, \infty) \times [0, 2\pi] \to \mathbb{R}$ be defined by $f(x, y) = \frac{1}{x}\sin(y)$. Compute

$$\int_1^\infty \int_0^{2\pi} f(x, y)\, dy\, dx.$$

What happens when considering

$$\int_0^{2\pi} \int_1^\infty f(x, y)\, dx\, dy?$$