Problem set 18: analysis in $\mathbb{R}^n$.

(1) Give an example of two functions $f, g\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\lim_{x\to 0} f(x)$ and $\lim_{y\to f(0)} g(y)$ exist but $\lim_{x\to 0} g \circ f(x) \neq g(\lim_{x\to 0} f(x))$.

(2) Give a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $\lim_{x\to 0}\lim_{y\to 0} f(x,y) \neq \lim_{y\to 0}\lim_{x\to 0} f(x,y)$.

(3) Show that $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \frac{x^2 y}{x^4 + y^2}$$

for $(x,y) \neq (0,0)$ and $f(0,0) = (0,0)$ is not continuous.

(4) Show that $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x) = \sin\left(\frac{1}{\|x\|}\right)$ for $x \neq 0$ and $f(0) = 0$ is not continuous.

(5) Let (X, d_X) and (Y, d_Y) be metric space. Suppose that X is such that whenever it is contained in a family of open balls, then we can find a finite subfamily of this family that still covers X (that means that X is *compact*). Show that if $f\colon X \to Y$ is continuous, then it is also uniformly continuous.

(6) Show that $f\colon (0,\infty) \to (0,\infty)$ defined by $f(x) = \frac{1}{x}$ is not uniformly continuous.

(7) Make a contour plot of $f(x,y) = x^2 + y^2$.

(8) Let (X, d_X) and (Y, d_Y) be metric spaces. Show that f is continuous at $x_0 \in X$ if and only if $\lim_{n\to\infty} f(x_n) = f(x_0)$ for all sequences $(x_n)_n$ with $\lim_{n\to\infty} x_n = x_0$.

(9) Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f\colon X \to Y$ be a map, $x_0 \in X$, and $y_0 \in Y$. Further, let $(x_n)_n$ be a sequence in X converging to x_0. Suppose $(y_n)_n$ and $(z_n)_n$ are subsequences of $(x_n)_n$ such that $\bigcup_{n\in\mathbb{N}}\{y_n, z_n\} = \bigcup_{n\in\mathbb{N}}\{x_n\}$. Show that if $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f(z_n) = y_0$, then also $\lim_{n\to\infty} f(x_n) = y_0$.

(10) Let (X, d_X) and (Y, d_Y) be metric spaces and $f\colon X \to Y$ a map. Suppose that $x_0 \in X$ and $A \subset X$ is such that $f|_{A\cup\{x_0\}}\colon A \cup \{x_0\} \to Y$ and $f|_{X\setminus A\cup\{x_0\}}\colon X \setminus A \cup \{x_0\} \to Y$ are continuous at x_0 (by $f|_B$ we mean the restriction of f to a set B, that is $f|_B$ is defined on B and $f|_B(x) = f(x)$ for all $x \in B$). Show that f is continuous at x_0 as well.

(11) Give an example of two continuous functions $f, g\colon \mathbb{R}^2 \to \mathbb{R}$ and a point $x_0 \in \mathbb{R}^2$ such that f is continuous at x_0, g is not, but fg is.

(12) Give a sequence of functions $(f_n)_n$, each $f_n\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous, $f(x) := \lim_{n\to\infty} f_n(x)$ exists for all $x \in \mathbb{R}^2$, but f is not continuous.

(13) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function with $f(0) = 0$ and $f(1,1) = 2$. Is there some $x \in \mathbb{R}^2$ with $f(x) = 1$?

(14) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be continuous with $f(0) = 0$ and $g\colon \mathbb{R}^2 \to \mathbb{R}$ be such that there is some $M \in \mathbb{R}$ with $|g(x)| \leq M$ for all $x \in \mathbb{R}^2$. Show that fg is continuous at 0.

(15) Show that $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $\sqrt{x^2 + x^2}\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)$ and $f(0) = 0$ is continuous at 0.

(16) Give an example of a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ that is continuous only at 0.

(17) Let (X, d_X) be a metric space and $f\colon X \to \mathbb{R}^n$ be a map, where n is a positive natural number. Writing $f = \sum_{k=1}^{n} f_k \cdot e_k$, show that f is continuous if and only if each f_k is.

(18) Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be linear. Show that it is continuous.

(19) Let (X, d_X) and (Y, d_Y) be metric spaces. Assume that $(f_n)_n$ is a sequence of continuous functions $f_n\colon X \to Y$ such that $(f_n(x))_n$ converges to some number $f(x)$ for each $x \in X$. We assume further that for all $\varepsilon > 0$ there is some $N \in \mathbb{N}$

such that $d_Y(f(x), f_n(x)) < \varepsilon$ for all $n \geq N$ and all $x \in X$. Show that f is continuous.

(20) Show that $f\colon \mathbb{R} \to \mathbb{R}$ defined by $\sum_{n=1}^{\infty} \frac{1}{2^n} \sin(x)$ is a continuous function.