Problem set 14: theorems about derivatives.

(1) Assume that $f: \mathbb{R} \to \mathbb{R}$ is differentiable and $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x)$. Show that there is some x with $f'(x) = 0$.

(2) Find a function $f: \mathbb{R} \to \mathbb{R}$ and distinct $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$ but there is no $x \in \mathbb{R}$ with $f'(x) = 0$.

(3) Let $f: \mathbb{R} \to \mathbb{R}$ be differentiable. Suppose that there is some $h \in \mathbb{R}$ such that $f(x + h) = f(x)$ for all $x \in \mathbb{R}$. Show that there is some $x_0 \in \mathbb{R}$ with $f'(x_0) = 0$.

(4) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 \sin(x)$. Show that there is a point $x_* \in \mathbb{R}$ such that $f''(x_*) = 0$.

(5) Find all local minima of $f: [0, \infty) \to \mathbb{R}$ defined as $f(x) = (x - 3)^3 + 5$.

(6) Assume that $f: (a, b) \to \mathbb{R}$ is twice differentiable, where $a, b \in \mathbb{R}$ with $a < b$. Show that if f'' is constant 0, then $f(x) = \alpha x + \beta$ for some real α and β.

(7) Find a nonconstant function $f: \mathbb{R} \to \mathbb{R}$ that attains a local minimum in uncountable many points.

(8) Assume that $f, g: \mathbb{R} \to \mathbb{R}$ are both convex. Is $g \circ f$ convex as well?

(9) Assume that $f, g: \mathbb{R} \to \mathbb{R}$ are both convex. Is fg convex as well?

(10) Compute $\lim_{x\to0} \frac{e^x - e^{-x}}{\sin(x)}$.

(11) Compute $\lim_{x\to1+} \frac{\log x}{\sqrt{x^2-1}}$.

(12) Compute $\lim_{x\to0} \frac{3^x - 2^x}{x}$.

(13) Compute $\lim_{x\to\frac{\pi}{2}} \frac{1-\sin(x)+\cos(x)+x-\frac{\pi}{2}}{\sin(2x)-\cos(x)}$.

(14) Compute $\lim_{x\to\infty} \frac{\log(\log(\log(x)))}{x}$.

(15) Compute $\lim_{x\to\infty} \left(1 - \frac{a}{x}\right)^x$, where a is a real number.

(16) What is $\lim_{x\to\infty} x^{1/x}$?

(17) Compute $\lim_{x\to1} \left(\frac{1}{\log x} - \frac{x}{\log x}\right)$.

(18) Calculate $\lim_{x\to0} \left(\frac{1}{x^2} - \frac{1}{\sin(x)}\right)$.