Problem set 12: limits and continuity.

(1) Determine the following limit

$$\lim_{x\to 3} \frac{x^3+8}{x-4}.$$

(2) Determine the following limit

$$\lim_{x\to 3} \frac{x^2-1}{x-3}.$$

(3) Calculate

$$\lim_{x\to 2} \frac{8-x^3}{x-2}.$$

(4) Calculate

$$\lim_{x\to 4} \frac{(x-4)(-1)^{[x]}}{x^2-16}$$

in case it exists.

(5) Let n be a natural number. Compute, if it exists,

$$\lim_{x\to 1} \frac{2x^n-2}{x-1}.$$

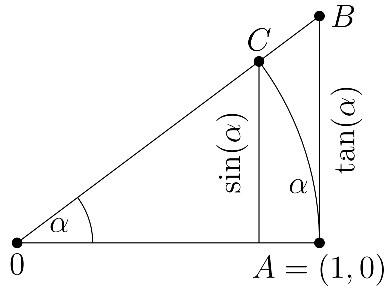
(6) Compute the following limit if it exists

$$\lim_{x\to 7} \frac{\sqrt{x}-7}{x^2-49}.$$

(7) Calculate the following limit in case it exists:

$$\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{\sqrt{x^2+36}-6}.$$

(8) (a) Compare the areas of the triangles $0AB$, $0AC$, and of the circular wedge to conclude that $\sin x < x < \tan x$ for $0 < x < \pi/2$.



(b) Show that $\lim_{x\to 0} \sin(x) = 0$.
(c) Show that $\lim_{x\to 0} \cos(x) = 1$.
(d) Show that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

(9) Compute the following limit in case it exists

$$\lim_{x\to 0} \frac{5x}{6\sin(3x)}.$$

(10) Calculate the following limit if it exists.

$$\lim_{x\to\infty} 2\sin x.$$

(11) Does the following limit exist? If so, what is its value?

$$\lim_{x\to\infty} \frac{\sin(2x)}{x^2}.$$

(12) Is there a function $f\colon \to \mathbb{R}$ such that for every $z \in [0,1]$, there is a sequence $(x_n)_n$ of real numbers converging to 0 such that

$$\lim_{n\to\infty} f(x_n) = z?$$

(13) What are the one-sided limits of $x \cdot \left[\frac{1}{x}\right]$ at 0?

(14) Investigate the left- and right-hand limit of $xe^{2/x}$ at 0.

(15) Find a function $f\colon \mathbb{R} \to \mathbb{R}$ that is only continuous at 0.

(16) Assume that $f\colon (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is defined by $f(x) = \frac{\cos^2(x)}{1-\sin(x)}$. Is there a continuous function $F\colon [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$ with $F(x) = f(x)$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$?

(17) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x - [x]$. Is it continuous?

(18) Determine the points where $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = [x] + [-x]$ is continuous.

(19) Does $f\colon [0,1] \setminus \{0\} \to \mathbb{R}$ defined as

$$f(x) = \frac{x}{\log(1+x)}$$

have a continuous extension to $[0,1]$?

(20) Is $f\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{x}{\log(x)}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

continuous?

(21) Is $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ x \sin\left(\frac{1}{x}\right), & \text{otherwise.} \end{cases}$$

continuous?

(22) Does $f\colon [-1,1] \setminus \{0\} \to \mathbb{R}$ defined by

$$f(x) = \frac{x}{\sqrt{\sin(x)}}$$

have a continuous extension to $[-1,1]$?

(23) Does $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by

$$f(x) = \frac{e^{\frac{1}{x}-2}}{e^{\frac{1}{x}+3}}$$

have a continuous extension to $\mathbb{R}$?