Problem set 11: sequences and series.

(1) Calculate the limit (as n tends to infinity) of the following sequences:
(a) $u_n = \frac{n+1}{n+2}$;
(b) $u_n = \frac{7n-5}{6-6n}$;
(c) $u_n = \frac{n^2}{7-n^4}$;
(d) $u_n = \frac{(-0.4)^n}{n}$.

(2) Does $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n-1}\right)$ exist? If so, calculate it.

(3) Show that

$$\lim_{n\to\infty}\frac{n^2}{2^n}$$

exists and calculate its value.

(4) Calculate the following limit in case it exists:

$$\lim_{n\to\infty}\frac{3^{2n-1}+7}{9^n+5}.$$

(5) Calculate the following limit in case it exists:

$$\lim_{n\to\infty}\sqrt[n]{5^n+6^n+7^n}.$$

(6) Calculate

$$\lim_{n\to\infty}\frac{\sum_{k=1}^n k}{(n+1)^2}.$$

(7) What is the following limit

$$\lim_{n\to\infty}\frac{\sum_{k=0}^n \frac{1}{5^k}}{\sum_{k=0}^n \frac{1}{4^k}}?$$

(8) Calculate

$$\lim_{n\to\infty}\frac{\log_3 n^8}{\log_9 n}.$$

(9) Compute

$$\sum_{n=1}^{\infty}\frac{2}{n(n+1)}.$$

Hint: Search for a and b such that

$$\frac{a}{n}+\frac{b}{n+1}=\frac{2}{n(n+1)}.$$

(10) Does the following series converge

$$\sum_{n=1}^{\infty}\frac{1}{5n-2}?$$

(11) Does the following series converge

$$\sum_{n=1}^{\infty}\frac{\log n}{n^4}?$$

(12) Does the following series converge

$$\sum_{n=1}^{\infty} \frac{\cos(4^n)}{3^n}?$$

(13) Does the following series converge

$$\sum_{n=1}^{\infty} \frac{1}{n} \cos\left(\frac{\pi}{n}\right)?$$

(14) Does the following series converge

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}?$$

(15) Let $(a_n)_n$ and $(b_n)_n$ be sequences of real numbers. Show that for all $N \in \mathbb{N}$

$$a_1 b_1 + \sum_{n=2}^{N} a_n(b_n - b_{n-1}) = \sum_{n=1}^{N-1} (a_n - a_{n+1}) b_n + a_N b_N.$$

(16) Assume that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge for some sequences $(a_n)_n$ and $(b_n)_n$ of real numbers. Is it also true that

$$\sum_{n=1}^{\infty} a_n b_n$$

converges?

(17) Give an example of a sequence $(a_n)_n$ such that $a_n > 1$ for all $n \in \mathbb{N}$ and

$$\lim_{N \to \infty} \prod_{n=1}^{N} a_n$$

exists.