**Problem set 10: orthogonal matrices, classification of real and complex orthogonal spaces, adjoint and self-adjoint maps.**

(1) Let $R(v_1, v_2, \ldots, v_n)$ denote the parallelepiped spanned by the vectors $(v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$. Find matrices with respect to the canonical bases of isometries of the following sets:
(a) the square $\begin{pmatrix} -1 \\ -1 \end{pmatrix} + R(2\varepsilon_1, 2\varepsilon_2)$ in $\mathbb{R}^2$ with the usual dot product;
(b) the square $\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + R(2\varepsilon_1, 2\varepsilon_2)$ in $\mathbb{R}^3$ with the usual dot product;
(c) the cube $\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + R(2\varepsilon_1, 2\varepsilon_2, 2\varepsilon_3)$ in $\mathbb{R}^3$ with the usual dot product.

(2) Represent as a composition of hyperplane symmetries the endomorphism of $\mathbb{R}^4$ with the usual dot product whose matrix in the canonical basis is equal to $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

(3) Check that the set $\mathrm{SO}(\mathbb{R}, 2)$ of real orthogonal matrices of degree 2 whose determinant is equal to 1 is given by $\left\{\begin{bmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{bmatrix} : x \in [0, 2\pi)\right\}$. Determine the set $\mathrm{O}(\mathbb{R}, 2)$ of real orthogonal matrices of degree 2.

(4) Consider the space $\mathbb{R}^2$ with the quadratic form $q\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 - y^2$. Check that $\mathrm{SO}(\mathbb{R}, q) = \left\{\pm\begin{bmatrix} \cosh(x) & \sinh(x) \\ -\sinh(x) & \cosh(x) \end{bmatrix} : x \in \mathbb{R}\right\}$, where

$$\begin{aligned} \sinh(x) &= \frac{e^x - e^{-x}}{2} \\ \cosh(x) &= \frac{e^x - e^{-x}}{2} \\ \mathrm{th}(x) &= \frac{\sinh(x)}{\cosh(x)}. \end{aligned}$$

Determine $\mathrm{O}(\mathbb{R}, q)$ .

(5) Pick the third column in the following matrix so that the resulting matrix is orthogonal, if:

$$A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & ? \\ \frac{1}{\sqrt{3}} & 0 & ? \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & ? \end{bmatrix}$$

(6) Check that the orthogonal space $(\mathbb{R}^3, \xi)$, where $\xi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}\right) = xx' + xy' + x'y + 3yy' + zz'$ is Euclidean. Find the general equation of a line perpendicular to the plane $\mathrm{Sol}(X_1 = 0)$ and intersecting the lines $L_1, L_2$, where $L_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \mathrm{lin}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \mathrm{lin}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right)$.

(7) For what values of the parameter $a \in \mathbb{R}$ the orthogonal space $(\mathbb{R}^6, \xi)$, where

$$q\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}\right) = a({x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 + {x_5}^2 + {x_6}^2) + 2(x_1x_2 + x_1x_3 + x_1x_4 + x_1x_5 + x_1x_6$$
$$+ x_2x_3 + x_2x_4 + x_2x_5 + x_2x_6 + x_3x_4 + x_3x_5 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6)$$

is Euclidean?

(8) Which of the following matrices are similar over the field of real numbers, and which ones over the field of complex numbers?

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

(9) Find the signature of the space $(\mathbb{R}^3, \xi)$ where $q\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = yz + xz + xy$.

(10) Check if the matrices $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}$ are similar:

(a) over the real numbers;
(b) over the rational numbers;
(c) over the complex numbers.

(11) Depending on the parameters $a, b, c, d$ find the signature of $(\mathbb{R}^4, \xi)$, where

$$q\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = a^2{x_1}^2 + 2ax_1(x_2 + x_3 + x_4) + (b^2 + 1){x_2}^22 + (4b + 2)x_2x_3 + (6b + 2)x_2x_4$$
$$+ (c^2 + 5){x_3}^2 + (6c + 14)x_3x_4 + (d^2 + 19){x_4}^2.$$

(12) Depending on the parameter $a$ find the signature of the orthogonal space $(\mathbb{R}^3, \xi)$ if $q\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) =$

(a) $5{x_1}^2 + {x_2}^2 + a{x_3}^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3$;
(b) $2{x_1}^2 + {x_2}^2 + 3{x_3}^2 + 2ax_1x_2 + 2x_1x_3$;
(c) ${x_1}^2 + {x_2}^2 + 5{x_3}^2 + 2ax_1x_2 - 2x_1x_3 + 4x_2x_3$;

(d) ${x_1}^2 + 4{x_2}^2 + {x_3}^2 + 2ax_1x_2 + 10x_1x_3 + 6x_2x_3$;
(e) $-{x_1}^2 + a{x_3}^2 - {x_3}^2 + 4x_1x_2 + 8x_2x_3$;
(f) $a{x_1}^2 - 2{x_2}^2 - 3{x_3}^2 + 2x_1x_2 - 2x_1x_3 + 2x_2x_3$.

(13) For what values of the parameter $a \in \mathbb{R}$ the orthogonal spaces $(\mathbb{R}^3, \xi)$ and $(\mathbb{R}^3, \eta)$ are isometric, if:

(a) $\xi\left(\begin{bmatrix} p \\ q \\ r \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = px + qy - rz$, $q_\eta\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x^2 + 2xy - 2xz + (3-a)y^2 + (2-2a)yz + (4-2a)z^2$;

(b) $\xi\left(\begin{bmatrix} p \\ q \\ r \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = px - qy + rz$, $q_\eta\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x^2 + 2xy - 2xz + (3-a)y^2 + (2-2a)yz + (6-2a)z^2$;

(c) $\xi\left(\begin{bmatrix} p \\ q \\ r \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = px + qy + rz$, $q_\eta\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x^2 + 4xy + (6-a)y^2 + (12-6a)yz + (19-10a)z^2$;

(d) $\xi\left(\begin{bmatrix} p \\ q \\ r \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = px - qy - rz$, $q_\eta\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x^2 + 6xy + (11-a)y^2 + (8-4a)yz + (1-a)z^2$.

(14) Check that a hyperplane symmetry is self-adjoint.

(15) Check that an orthogonal projection is self-adjoint.

(16) Find an orthogonal matrix $C$ such that the matrix $C^\intercal AC$ is diagonal if $A =$

(a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$; (b) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$; (c) $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$; (d) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$;

(e) $\begin{bmatrix} 2 & -3 & -2 & 1 \\ -3 & 2 & -2 & -1 \\ -2 & -2 & 1 & 0 \\ 1 & -1 & 0 & 4 \end{bmatrix}$; (f) $\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$; (g) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; (h) $\begin{bmatrix} 1 + t\cos(\frac{2}{t}) & t\sin(\frac{2}{t}) \\ t\sin(\frac{2}{t}) & 1 - t\cos(\frac{2}{t}) \end{bmatrix}$.