Problem set 9: bilinear and quadratic forms, orthogonal bases.

(1) Check if the following maps $\xi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$:

(a) $\xi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}\right) = xx' + x^2y' + z'$; (b) $\xi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}\right) = xz' + yx' + 2$;

(c) $\xi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}\right) = xx' + 2yz' + zz'$; (d) $\xi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}\right) = xx' + xy' + z'$;

(e) $\xi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}\right) = 0$; (f) $\xi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}\right) = 1$.

are bilinear. Which of them are symmetric?

(2) In the orthogonal space $(\mathbb{Q}^3, \xi)$ the matrix of a bilinear map ξ in the basis

$$\mathcal{B} = \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

is equal to

(a) $\begin{bmatrix} 2 & 1 & -2 \\ 1 & 1 & -1 \\ -2 & -1 & 2 \end{bmatrix}$; (b) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & -1 \\ 3 & -1 & 2 \end{bmatrix}$; (c) $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 2 \end{bmatrix}$.

Find $\xi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$. Find an orthogonal basis of $(\mathbb{Q}^3, \xi)$.

(3) The bilinear map $\xi : \mathbb{Q}^4 \times \mathbb{Q}^4 \to \mathbb{Q}$ has the following matrix in the canonical basis $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ -1 & 0 & 1 & 4 \end{bmatrix}$$

Let $W = \operatorname{lin}(\varepsilon_1, \varepsilon_1 + \varepsilon_2)$. Find an orthogonal basis of the space W.

(4) Decompose the orthogonal space (K^3, ξ), where ξ has the following matrix in the basis $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$:

$$\begin{bmatrix} 2 & 1 & -2 \\ 1 & 1 & -1 \\ -2 & -1 & 2 \end{bmatrix},$$

as a direct orthogonal sum of a nondegenerate space and totally degenerate space.

(5) In the space ${\mathbb{Z}_5}^2$ consider the quadratic form given by

$$q\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 + y^2.$$

(a) Find the orthogonal completion $\begin{bmatrix} 2 \\ 1 \end{bmatrix}^\perp$ of the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(b) Find the orthogonal completion of the space $\operatorname{lin}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$.

(6) In the space ${\mathbb{Z}_2}^2$ consider the bilinear form ξ given by

$$\xi\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right) = \begin{vmatrix} a & c \\ b & d \end{vmatrix}.$$

Show that $({\mathbb{Z}_2}^2, \xi)$ is a nondegenerate orthogonal space, where every vector is isotropic. Show that this space has no orthogonal basis.

(7) In the space $\mathbb{R}^2$ consider a quadratic form given by

$$\text{i) } q\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 + 2xy + y^2 \qquad \text{ii) } q\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 + y^2.$$

(a) Find the orthogonal completion of the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(b) Find all orthogonal completions of the line $\operatorname{lin}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$.

(8) Find an orthogonal basis of $(\mathbb{Q}^3, \xi)$ if $q\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = yz + xz + xy$.

(9) Show that the space $\mathbb{R}^3$ with the bilinear map $\xi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by either one of the following formulae:

(a) $\xi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = x_1y_1 + x_2y_2 + x_2y_1 + x_1y_2 + x_3y_3;$

(b) $\xi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = (x_1 - x_2)(y_1 - y_2) + (x_1 - x_3)(y_1 - y_3) + x_3y_3;$

is an Euclidean space, while the space $\mathbb{R}^3$ with the bilinear map $\xi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by either one of the following formulae:

(c) $\xi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = x_1y_1 + x_2y_2;$

(d) $\xi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = x_2y_1 + x_1y_2 + x_3y_3;$

(e) $\xi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = x_1y_1 + x_2y_2 + x_3y_3 + x_2y_1 + x_1y_2 + x_3y_1 + x_1y_3$

is not Euclidean. Which of the above mentioned spaces are nondegenerate? For each of the spaces (c), (d) and (e) find a nonzero totally degenerate subspace.

(10) Show that the orthogonal spaces $({\mathbb{Z}_3}^3, \xi)$ and $({\mathbb{Z}_3}^3, \zeta)$ are isomorphic, where $q_\xi\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = a^2 + b^2 + c^2$ and $q_\zeta\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = a^2 - b^2 - c^2$.

(11) Check if the orthogonal spaces $(\mathbb{R}^2, \xi)$, $(\mathbb{R}^2, \zeta)$ where $q_\xi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 - y^2$, $q_\zeta\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 + 2xy + y^2$ are isomorphic.

(12) Show that the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ are similar over the fields $\mathbb{Z}_7$ and $\mathbb{R}$, but are not similar over the fields $\mathbb{Z}_3$ and $\mathbb{Q}$.

(13) Are matrices $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}$ similar

(a) over the field of real numbers? b) over the field of rational numbers?

(14) Find an isomorphism of the orthogonal spaces (V, ξ) and (W, ζ) over the field $\mathbb{Z}_3$ or show that they are not isomorphic:

(a) ξ and ζ with respect to certain bases have the matrices $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, respectively;

(b) ξ is the usual dot product in $V = \mathbb{Z}_3^4$ and $q_\zeta(xw_1 + yw_2 + zw_3 + tw_4) = xy + zt$ for a certain basis (w_1, w_2, w_3, w_4) of W.

(15) In the orthogonal space $(\mathbb{R}^3, \xi)$ a quadratic form is given by $q_\xi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x^2 - 2xy + 3y^2 + z^2$.

(a) Find the matrix of the orthogonal projection on the plane $\operatorname{lin}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right)$ in the basis $(\varepsilon_1, \varepsilon_1 + \varepsilon_2, \varepsilon_3)$.

(b) Find the matrix of the orthogonal symmetry with respect to the plane $\operatorname{Sol}(X - Y + Z = 0)$ in the canonical basis.

(c) Find a formula for the orthogonal symmetry with respect to the line $\operatorname{Sol}\left(\begin{cases} X + 2Y - Z = 0 \\ 2X + Y = 0 \end{cases}\right)$.

(16) Find the orthogonal projection of the vector v on the subspace L in the Euclidean space $\mathbb{R}^4$ with the usual dot product, if

(a) $v = \begin{bmatrix} 4 \\ -1 \\ -3 \\ 4 \end{bmatrix}$, $L = \operatorname{lin}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}\right)$;

(b) $v = \begin{bmatrix} 7 \\ -4 \\ -1 \\ 2 \end{bmatrix}$, $L = \operatorname{Sol}\left(\begin{cases} 2x_1 + x_2 + x_3 + 3x_4 = 0 \\ 3x_1 + 2x_2 + 2x_3 + x_4 = 0 \\ x_1 + 2x_2 + 2x_3 - 4x_4 = 0 \end{cases}\right)$.

(17) Give matrices with respect to the canonical basis of two different axial symmetries which map $\operatorname{lin}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ onto $\operatorname{lin}\left(\begin{bmatrix} -1 \\ 7 \end{bmatrix}\right)$ in the space $\mathbb{R}^2$ with the usual dot product.

(18) In the space $\mathbb{R}^3$ with the usual dot product the following planes are given: $\operatorname{lin}\left(\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right)$ and $\operatorname{lin}\left(\begin{bmatrix}1\\3\\-2\end{bmatrix},\begin{bmatrix}1\\-2\\3\end{bmatrix}\right)$. Give matrices with respect to the canonical bases of two different plane symmetries which map one plane onto the other.

(19) Find an orthogonal basis of the orthogonal space (V,ξ), if:
(a) $V=\mathbb{R}^3$, and the matrix of the bilinear map $\xi:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}$ in the basis $(\varepsilon_1+\varepsilon_2+\varepsilon_3,\varepsilon_1+\varepsilon_3,\varepsilon_3)$ is equal to $\begin{bmatrix}-1&0&1\\0&1&1\\1&1&1\end{bmatrix}$;
(b) $V=\mathbb{R}^4$, and $q_\xi\left(\begin{bmatrix}x\\y\\z\\t\end{bmatrix}\right)=2xz+yz-xy$;
(c) $V={\mathbb{Z}_5}^3$, and the matrix of ξ in the canonical basis is equal to $\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$.

(20) Find a normed orthogonal basis of the space $(\mathbb{Q}^2,\xi)$ where $q\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=2x^2+2y^2$. Does there exist a normed orthogonal basis of the space $(\mathbb{Q}^2,\xi)$, where $q\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=7x^2+7y^2$?

(21) Using either the Jacobi or Lagrange algorithm find a matrix of the bilinear map in a certain orthogonal basis, if:
(a) $F(x_1,x_2)={x_1}^2+x_1x_2+{x_2}^2$; (b) $F(x_1,x_2,x_3)=x_1x_2+x_2x_3+x_1x_3$
(c) $F(x_1,x_2,x_3)=99{x_1}^2-12x_1x_2+48x_1x_3+130{x_2}^2-60x_2x_3+71{x_3}^2$;
(d) $F(x_1,x_2,x_3,x_4)={x_1}^2+4{x_2}^2+8{x_3}^2-{x_4}^2-4x_1x_2+6x_1x_3-12x_2x_3+2x_3x_4$;
(e) $F(x_1,x_2,x_3,x_4,x_5)={x_1}^2+4{x_2}^2+8{x_3}^2-{x_4}^2-4x_1x_2+6x_1x_3-12x_2x_3+2x_3x_4+x_2x_5-x_4x_5$.

(22) Find a diagonal matrix similar to the matrix
(a) $\begin{bmatrix}1&1&0\\1&1&1\\0&1&0\end{bmatrix}$; (b) $\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix}$
over a field F.

(23) Find at least one nonsingular matrix $P\in\mathrm{GL}_n(\mathbb{Q})$ such that the matrix $P^\intercal AP$ is diagonal, if
(a) $A=\begin{bmatrix}1&1&0\\1&2&2\\0&2&3\end{bmatrix}$; (b) $A=\begin{bmatrix}1&1&1\\1&5&-1\\1&-1&5\end{bmatrix}$; (c) $A=\begin{bmatrix}1&1\\1&1\end{bmatrix}$;
(d) $A=\begin{bmatrix}2&1&1&0\\1&2&0&1\\1&0&2&1\\0&1&1&2\end{bmatrix}$; (e) $A=\begin{bmatrix}2&1\\1&2\end{bmatrix}$.

(24) Find at least one nonsingular matrix $P \in \mathrm{GL}_n(\mathbb{R})$ such that $P^\intercal P = A$, if

(a) $A = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$; (b) $A = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$; (c) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$, (d) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix}$.

Is there a solution such that $P \in \mathrm{GL}_3(\mathbb{Q})$?

(25) Find at least one solution $X \in \mathrm{GL}_4(\mathbb{R})$ of the equation $X^\intercal X = \begin{bmatrix} 1 & 2 & -5 & -5 \\ 2 & 5 & -12 & -13 \\ -5 & -12 & 30 & 33 \\ -5 & -13 & 33 & 39 \end{bmatrix}$.