Problem set 8: linear maps, matrices of linear maps, eigenvectors and eigenvalues, diagonalization.

(1) Which of the following maps $\varphi : K^n \to K^m$ are linear, if:

(a) $n = m = 3$, $\varphi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+z \\ 2x+z \\ 3x-y+z \end{bmatrix}$; (b) $n = m = 3$, $\varphi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y+1 \\ z+2 \end{bmatrix}$;

(c) $n = m = 3$, $\varphi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x+y \\ x+z \\ z \end{bmatrix}$; (d) $n = m = 3$, $\varphi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-y+z \\ z \\ y \end{bmatrix}$;

(e) $n = 4$, $m = 3$, $\varphi\left(\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}\right) = \begin{bmatrix} x-y+2t \\ 2x+3y+5z-t \\ x+z-t \end{bmatrix}$;

(f) $n = 4$, $m = 3$, $\varphi\left(\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}\right) = \begin{bmatrix} x-y+2t \\ 2x-3y+5z-t \\ x-z-t \end{bmatrix}$;

(g) $n = m = 4$, $\varphi\left(\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}\right) = \begin{bmatrix} x+3y-2t \\ x+y+z \\ 2y+t \\ y+z \end{bmatrix}$;

(h) $n = m = 4$, $\varphi\left(\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}\right) = \begin{bmatrix} x+3y-2t \\ x+y+z \\ 2y-3t \\ 2x+4y+z-2t \end{bmatrix}$;

(i) $n = m = 3$, $\varphi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+z \\ 2xz \\ 3x-y+z \end{bmatrix}$.

If φ is a linear map, check if it is a monomorphism, or an epimorphism.

(2) Let V, V_1, V_2, W be vector spaces and let $V = V_1 \oplus V_2$. Show that for each pair of linear maps $\varphi_i\colon V_i \to W$, $i = 1, 2$, there exists exactly one linear map $\varphi\colon V \to W$ such that $\varphi\,|_{V_i} = \varphi_i$. If $V = W$ and $\varphi_1 = \mathrm{Id}_{V_1}, \varphi_2 = -\mathrm{Id}_{V_2}$ then φ will be called the *symmetry* of V_1 along V_2. If $\varphi_1 = \mathrm{Id}_{V_1}$, and φ_2 is the zero map, then φ will be called the *projection* of V onto V_1 along V_2.

(3) Find kernels and images of linear maps from Problem (1).

(4) Find kernels and images of the symmetry (projection) of V_1 (onto V_1) along V_2 (see Problem (**??**)).

(5) A linear map $\varphi : K^2 \to K^3$ is given by $\varphi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+3y \\ x-y \\ 3y \end{bmatrix}$. Find:

(a) images of the following subspaces: K^2, $\mathrm{lin}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$, $\mathrm{lin}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$, $\mathrm{lin}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$, $\left\{\begin{bmatrix} x \\ y \end{bmatrix} \in K^2 : 2x+3y = 0\right\}$;

(b) counterimages of the following subspaces: K^3, $\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$, $\operatorname{lin}\left(\begin{bmatrix}2\\1\\3\end{bmatrix}\right)$, $\operatorname{lin}\left(\begin{bmatrix}2\\1\\0\end{bmatrix}\right)$, $\operatorname{lin}\left(\begin{bmatrix}3\\-1\\3\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right)$, $\left\{\begin{bmatrix}x\\y\\z\end{bmatrix} \in K^3 : x+y+z=0\right\}$.

(6) A linear map $\varphi : V \to W$ satisfies the following conditions

$$\begin{aligned}\varphi(\alpha_1) &= \beta_1 + 2\beta_2 + 3\beta_3,\\ \varphi(\alpha_2) &= 4\beta_1 + 5\beta_2 + 6\beta_3,\\ \varphi(\alpha_3) &= 7\beta_1 + 8\beta_2 + 9\beta_3\end{aligned}$$

where $(\alpha_1, \alpha_2, \alpha_3)$ is a basis of V, and $(\beta_1, \beta_2, \beta_3)$ a basis of W. Find the dimensions of the image and of the kernel of φ.

(7) Is there a linear map $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ such that

(a) $\varphi\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\varphi\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\varphi\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\1\end{bmatrix}$, $\varphi\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\\1\end{bmatrix}$;

(b) $\varphi\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\varphi\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}3\\2\\1\end{bmatrix}$, $\varphi\left(\begin{bmatrix}1\\2\\1\end{bmatrix}\right) = \begin{bmatrix}4\\4\\4\end{bmatrix}$;

(c) $\varphi\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\varphi\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}3\\2\\1\end{bmatrix}$, $\varphi\left(\begin{bmatrix}1\\-2\\1\end{bmatrix}\right) = \begin{bmatrix}4\\4\\4\end{bmatrix}$;

(d) $\varphi\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\0\end{bmatrix}$, $\varphi\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}3\\0\\1\end{bmatrix}$?

If so, discuss the number of solutions and find at least one such map.

(8) Find a linear map $\tau : \mathbb{R}^3 \to \mathbb{R}^3$ such that:

$$\tau\left(\begin{bmatrix}1\\1\\2\end{bmatrix}\right) = \begin{bmatrix}2\\1\\1\end{bmatrix},\ \tau\left(\begin{bmatrix}2\\1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\\2\end{bmatrix},\ \tau \circ \tau = \mathrm{id}_{\mathbb{R}^3}.$$

(9) Find:

(a) the symmetry in $\mathbb{R}^2$ of $\operatorname{lin}\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$ along $\operatorname{lin}\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$;

(b) the symmetry in $\mathbb{R}^3$ of $\operatorname{lin}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\right)$ along $\operatorname{lin}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$;

(c) the projection of $\mathbb{R}^2$ onto $\operatorname{lin}\left(\begin{bmatrix}2\\3\end{bmatrix}\right)$ along $\operatorname{lin}\left(\begin{bmatrix}-1\\1\end{bmatrix}\right)$;

(d) the projection of $\mathbb{R}^3$ onto $\operatorname{lin}\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$ along $\operatorname{lin}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\2\end{bmatrix}\right)$.

(10) Find a linear map $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\operatorname{Ker}\psi = \operatorname{lin}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right)$ and $\operatorname{Im}\psi = \operatorname{lin}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$.

How many solutions are there?

(11) In the vector space K^3 consider the bases $\mathcal{A}_3 = \left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right)$ and $\mathcal{B}_3 = \left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right)$, and in the vector space K^4 consider the bases $\mathcal{A}_4 = \left(\begin{bmatrix}2\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\2\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\0\\0\end{bmatrix}\right)$ and $\mathcal{B}_4 = \left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right)$. Find the matrix of a linear map $\varphi : K^n \to K^m$ in the bases $\mathcal{A}_n$ and $\mathcal{B}_m$ ($\mathcal{A}_n$ and $\mathcal{A}_m$; $\mathcal{B}_n$ and $\mathcal{B}_m$; $\mathcal{B}_n$ and $\mathcal{A}_m$), if:

(a) $n = m = 3$, $\varphi\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x+z\\2x+z\\3x-y+z\end{bmatrix}$; (b) $n = m = 3$, $\varphi\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x-y+z\\y\\z\end{bmatrix}$;

(c) $n = 4, m = 3$, $\varphi\left(\begin{bmatrix}x\\y\\z\\t\end{bmatrix}\right) = \begin{bmatrix}x-y+2t\\2x+3y+5z-t\\x+z-t\end{bmatrix}$;

(d) $n = 4, m = 3$, $\varphi\left(\begin{bmatrix}x\\y\\z\\t\end{bmatrix}\right) = \begin{bmatrix}x-y+2t\\2x-3y+5z-t\\x-z-t\end{bmatrix}$;

(e) $n = 3, m = 4$, $\varphi\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x+3y-2z\\x+y+z\\2y\\y+z\end{bmatrix}$; (f) $n = 3, m = 4$, $\varphi\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x+3y-2z\\x+y+z\\2y-3z\\2x+4y+z\end{bmatrix}$.

(12) Let $\varphi : K^3 \to V_1$ be the projection and $\psi : K^3 \to K^3$ the symmetry of V_1 along V_2, where:

(a) $V_1 = \operatorname{lin}(\varepsilon_1, \varepsilon_2)$, $V_2 = \operatorname{lin}(\varepsilon_1 + \varepsilon_3)$;

(b) $V_1 = \operatorname{lin}(\varepsilon_1, \varepsilon_2)$, $V_2 = \operatorname{lin}(\varepsilon_2 + \varepsilon_3)$;

(c) $V_1 = \operatorname{lin}(\varepsilon_1 + \varepsilon_2, \varepsilon_2)$, $V_2 = \operatorname{lin}(\varepsilon_1 + \varepsilon_3)$.

Find the matrix of φ in the bases $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ of K^3 and $(\varepsilon_1, \varepsilon_2)$ of V_1. Find the matrix of ψ in the bases $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and $(\varepsilon_1, \varepsilon_2, \varepsilon_1 + \varepsilon_3)$ of K^3.

(13) A linear map $\varphi : K^2 \to K^3$ has the following matrix in the bases $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$ and $\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\right)$:

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 3 & -2 \end{bmatrix}.$$

Find $\varphi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$.

(14) An endomorphism ψ of K^3 has the following matrix in the basis $(\varepsilon_1, \varepsilon_2, \varepsilon_1 + \varepsilon_3)$:

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix}.$$

Find ψ.

(15) An endomorphism ψ of $\mathbb{R}^3$ has the following matrix in the basis $(\varepsilon_1 - \varepsilon_2, \varepsilon_2, \varepsilon_1 + \varepsilon_3)$:

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix}.$$

Find bases of the kernel and of the image of ψ. Does the vector $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ belong to the kernel of ψ? What is the image of the vector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$?

(16) Consider the vector space $\mathbb{R}^n$ and its bases $\mathcal{A}$ and $\mathcal{B}$. Denote by $\mathcal{E}$ the canonical basis $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$. Find the transition matrices from $\mathcal{E}$ to $\mathcal{A}$, from $\mathcal{E}$ to $\mathcal{B}$, from $\mathcal{A}$ to $\mathcal{E}$ and from $\mathcal{A}$ to $\mathcal{B}$, if:

(a) $n = 2$, $\mathcal{A} = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \end{bmatrix}\right)$, $\mathcal{B} = \left(\begin{bmatrix} -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix}\right)$;

(b) $n = 3$, $\mathcal{A} = \left(\begin{bmatrix} 8 \\ -6 \\ 7 \end{bmatrix}, \begin{bmatrix} -16 \\ 7 \\ -13 \end{bmatrix}, \begin{bmatrix} 9 \\ -3 \\ 7 \end{bmatrix}\right)$, $\mathcal{B} = \left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\right)$;

(c) $n = 4$, $\mathcal{A} = \left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$, $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$.

(17) Let $\mathcal{A} = (\alpha_1, \alpha_2, \alpha_3)$, $\mathcal{B} = (\beta_1, \beta_2, \beta_3)$ be bases of the space $\mathbb{C}^3$. Find the matrix of the symmetry of $V_1 = \mathrm{lin}(\alpha_1, \alpha_2)$ along $V_2 = \mathrm{lin}(\alpha_3)$ in the basis $\mathcal{B}$, if $\alpha_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\beta_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \beta_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \beta_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Find the matrix of the projection onto V_1 along V_2.

(18) Find the coordinates of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ in the basis $\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right)$ of the space K^4 if $\mathrm{char}\, K \neq 2, 3$.

(19) Find formulae for the change of coordinates while passing from the basis

$$\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right)$$

to the basis

$$\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right)$$

of the space K^4 if $\mathrm{char}\, K \neq 2$.

(20) Find the matrix of $\varphi : K^3 \to K^3$ in the basis $(\varepsilon_1, \varepsilon_2 + \varepsilon_3, \varepsilon_1 + \varepsilon_2)$ if the matrix of φ in the basis

(a) $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, (b) $(\varepsilon_1 + \varepsilon_2, \varepsilon_2, \varepsilon_3)$

is equal to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(21) An endomorphism $\varphi \in \mathrm{End}(\mathbb{C}^2)$ has the following matrix in the basis $\mathcal{A} = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ i \end{bmatrix} \right)$:

(a) $\begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}$; (b) $\begin{bmatrix} 2 & 1 \\ -1 & 5 \end{bmatrix}$.

Find eigenvalues and eigenvectors of φ. What will be the solution if we assume that $\mathcal{A}$ is the canonical basis? And if we assume that $\varphi \in \mathrm{End}(\mathbb{R}^2)$?

(22) A is the matrix of an endomorphism $\varphi \in \mathrm{End}(\mathbb{C}^n)$ in the canonical basis. Find eigenvalues and eigenvectors of φ. If possible, find a basis of $\mathbb{C}^n$ consisting of eigenvectors of φ, as well as a matrix $C \in GL(n, \mathbb{C})$ such that the matrix $C^{-1}AC$ is diagonal.

$n = 2:$ (a) $A = \begin{bmatrix} 0 & 2 \\ -3 & 5 \end{bmatrix}$; (b) $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$; (c) $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$; (d) $A = \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}$;

$n = 3:$ (e) $A = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 3 \\ -1 & -3 & 0 \end{bmatrix}$; (f) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$; (g) $A = \begin{bmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ 4 & -8 & -2 \end{bmatrix}$;

$n=4:$ (h) $A=\begin{bmatrix}0&0&0&1\\0&0&2&0\\0&3&0&0\\4&0&0&0\end{bmatrix}$; (i) $A=\begin{bmatrix}0&0&0&0\\0&0&1&0\\0&2&0&0\\3&0&0&0\end{bmatrix}$; (j) $A=\begin{bmatrix}1&1&2&3\\0&2&2&4\\0&0&1&-2\\0&0&0&2\end{bmatrix}$;

(k) $A=\begin{bmatrix}1&1&2&3\\0&1&1&2\\0&0&2&0\\0&0&0&2\end{bmatrix}$; (l) $A=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-6&1&7&-1\end{bmatrix}$; (m) $A=\begin{bmatrix}1&1&0&0\\3&0&1&0\\-1&0&0&1\\-2&0&0&0\end{bmatrix}$.

(23) Find the characteristic polynomial of an endomorphism, which in a certain basis has the following matrix:

(a) $\begin{bmatrix}-a_{n-1}&-a_{n-2}&\cdots&-a_1&-a_0\\1&0&\cdots&0&0\\0&1&\cdots&0&0\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\cdots&1&0\end{bmatrix}$; (b) $\begin{bmatrix}0&0&\cdots&0&-a_0\\1&0&\cdots&0&-a_1\\0&1&\cdots&0&-a_2\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\cdots&1&-a_{n-1}\end{bmatrix}$.

(24) Find eigenvalues and corresponding eigenvectors of endomorphisms of real vector spaces whose matrices in the canonical bases are equal to:

(a) $\begin{bmatrix}-3&4\\2&-1\end{bmatrix}$; (b) $\begin{bmatrix}1&1\\1&-1\end{bmatrix}$; (c) $\begin{bmatrix}1&2\\2&-2\end{bmatrix}$; (d) $\begin{bmatrix}2&4\\5&3\end{bmatrix}$;

(e) $\begin{bmatrix}5&6&-3\\-1&0&1\\1&2&-1\end{bmatrix}$; (f) $\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$; (g) $\begin{bmatrix}0&2&1\\-2&0&3\\-1&-3&0\end{bmatrix}$.

(25) Find eigenvalues and corresponding eigenvectors of endomorphisms of complex vector spaces whose matrices in the canonical bases are equal to:

(a) $\begin{bmatrix}-1&2i\\-2i&2\end{bmatrix}$; (b) $\begin{bmatrix}0&a\\-a&0\end{bmatrix}$ for $a\in\mathbb{R}$;

(c) $\begin{bmatrix}0&1&0&\cdots&0&0\\-1&0&1&\cdots&0&0\\0&-1&0&\cdots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\cdots&0&1\\0&0&0&\cdots&-1&0\end{bmatrix}$.

(26) Find a formula for A^n, if A equals to

(a) $\begin{bmatrix}1&2\\2&2\end{bmatrix}$; (b) $\begin{bmatrix}0&2\\-3&5\end{bmatrix}$; (c) $\begin{bmatrix}1&1\\-1&3\end{bmatrix}$; (d) $\begin{bmatrix}1&2\\2&-2\end{bmatrix}$.

(27) Find a formula for a_n, if

(a) $a_0=0$, $a_1=1$, $a_{n+2}=a_{n+1}+a_n$ (Fibonacci sequence);

(b) $\mathrm{a}_0=1$, $a_1=2$, $a_{n+2}=3a_n-2a_{n+1}$.