Problem set 7: bases and dimension.

(1) Show that the vectors $\alpha_1, \ldots, \alpha_n$ form a basis of the space $\mathbb{Q}^n$ and find the coordinates of the vector β in such a basis, if

(a) $n=3$; $\alpha_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\beta = \begin{bmatrix} 6 \\ 9 \\ 14 \end{bmatrix}$;

(b) $n=3$; $\alpha_1 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\beta = \begin{bmatrix} 6 \\ 2 \\ -7 \end{bmatrix}$;

(c) $n=4$; $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 4 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 1 \\ 3 \\ -1 \\ 0 \end{bmatrix}$, $\beta = \begin{bmatrix} 7 \\ 14 \\ -1 \\ 2 \end{bmatrix}$.

(2) Find bases of the subspaces of solutions of the following systems of linear equations (over $\mathbb{R}$):

(a) $\begin{cases} x_1 + 3x_2 + 2x_3 = 0 \\ 2x_1 - x_2 + 3x_3 = 0 \\ 3x_1 - 5x_2 + 4x_3 = 0 \end{cases}$; (b) $\begin{cases} x_1 + x_2 - 3x_4 = 0 \\ x_1 - x_2 + 2x_3 - x_4 = 0 \\ 4x_1 - 2x_2 + 6x_3 + 3x_4 = 0 \end{cases}$.

(3) Find a basis and the dimension of a given subspace $\mathrm{lin}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of the space $\mathbb{Q}^4$ if:

(a) $\alpha_1 = \begin{bmatrix} 5 \\ 2 \\ -3 \\ 1 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 4 \\ 1 \\ -2 \\ 3 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 3 \\ 4 \\ -1 \\ 2 \end{bmatrix}$;

(b) $\alpha_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 5 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 4 \\ -3 \\ 1 \\ 3 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 3 \\ -2 \\ 3 \\ 4 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 4 \\ -1 \\ 15 \\ 17 \end{bmatrix}$, $\alpha_5 = \begin{bmatrix} 7 \\ -6 \\ -7 \\ 0 \end{bmatrix}$;

(c) $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -4 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 2 \\ 3 \\ -4 \\ 1 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 2 \\ -5 \\ 8 \\ -3 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 5 \\ 26 \\ -9 \\ -12 \end{bmatrix}$, $\alpha_5 = \begin{bmatrix} 3 \\ -4 \\ 1 \\ 2 \end{bmatrix}$.

(4) Pick vectors to form a basis of the subspace $\mathrm{lin}(\alpha_1, \alpha_2, \ldots, a_n) \subset \mathbb{Z}_7^m$ among $\alpha_1, \alpha_2, \ldots, a_n$, if

(a) $m=4$, $n=3$, $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 3 \\ 6 \\ 0 \\ 0 \end{bmatrix}$;

(b) $m=4$, $n=4$, $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 4 \\ 5 \\ 6 \\ 0 \end{bmatrix}$;

(c) $m=4$, $n=5$, $\alpha_1 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 1 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 4 \\ 2 \\ 1 \\ 2 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 6 \\ 3 \\ 5 \\ 3 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\alpha_5 = \begin{bmatrix} 6 \\ 0 \\ 4 \\ 0 \end{bmatrix}$;

(d) $m = 3$, $n = 5$, $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$, $\alpha_5 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(5) Is it possible to find a basis of K^4 consisting of vectors of the form:

(a) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$; $x_1 + x_2 + x_3 + x_4 = 0$; (b) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$; $x_1 + x_2 + x_3 + x_4 = 1$?

(6) Find a basis for each of the subspaces of $\mathbb{R}^4$ listed below as well as a basis of the sum $U_i + U_j$ and the intersection $U_i \cap U_j$, if:

(a) $U_1 = \operatorname{lin}\left(\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \\ -1 \end{bmatrix}\right)$, $U_2 = \left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x_1 + x_2 - 2x_3 + x_4 = 0\right\}$;

(b) $U_1 = \operatorname{lin}\left(\begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 1 \\ -3 \end{bmatrix}\right)$, $U_2 = \operatorname{lin}\left(\begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \\ -3 \end{bmatrix}\right)$,

$U_3 = \left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x_1 - x_2 + x_3 + x_4 = 0\right\}$;

(c) $U_1 = \left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : 2x_1 - x_2 + x_3 - 2x_4 = 0\right\}$,

$U_2 = \operatorname{lin}\left(\begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 1 \\ 5 \end{bmatrix}\right)$;

(d) $U_1 = \operatorname{lin}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 5 \end{bmatrix}\right)$, $U_2 = \operatorname{lin}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right)$.