Problem set 6: vector spaces and subspaces, linear combinations.

(1) Check which of the following subsets of the space K^4 are subspaces, where K is an arbitrary field.
  (a) $U = \{[t, t+1, 0, 1] : t \in K\}$;
  (b) $U = \{[t, u, t+u, t-u] : t, u \in K\}$;
  (c) $U = \{[tu, u, t, 0] : t, u \in K\}$;
  (d) $U = \{[x, y, z, t] : x + y - z = 0\}$;
  (e) $U = \{[x, y, z, t] : xy = 0\}$;
  (f) $U = \{t[1, 0, 1, 0] + u[0, -1, 0, 1] : t, u \in K\}$.

(2) Check which of the following subsets of the space $\mathbb{R}^4$ are subspaces:
  (a) $U = \{[t, u, t+u, t-u] : t \leq u\}$;
  (b) $U = \{[t, u, t, 0] : tu \geq 0\}$;
  (c) $U = \{[x, y, z, t] : x, y, z, t \in \mathbb{Q}\}$.

(3) Let $\mathbb{R}^\infty$ be the space of sequences of elements of the field $\mathbb{R}$. Check which of the following subsets are subspaces:
  (a) $U_1 = \{[a_1, a_2, \ldots] : a_{i+1} = a_i + a_{i-1}$ for every $i = 2, 3, \ldots\}$;
  (b) $U_2 = \{[a_1, a_2, \ldots] : a_i = \frac{1}{2}(a_{i-1} + a_{i+1})$ for every $i = 2, 3, \ldots\}$;
  (c) the set of all sequences $[a_1, a_2, \ldots]$, whose entries are almost all zero;
  (d) the set of all bounded sequences.

(4) Let $A \subset \mathbb{R}$ be a nonempty set and let $V = \mathbb{R}^A$ be the space of functions $A \to \mathbb{R}$. Check which of the following subsets are subspaces:
  (a) the set of all even functions, if $A = \mathbb{R}$;
  (b) the set of all odd functions, if $A = \mathbb{R}$;
  (c) the set of all increasing functions;
  (d) the set of all monotone functions;
  (e) $U = \{f \in V : f(0) = f(1)\}$, if $A = [0, 1]$;
  (f) $U = \{f \in V : f(x) = 0$ for every $x \in B\}$, if $B \subset A$ and $B \neq A$.

(5) Show that if $U_1 = \mathrm{lin}(\alpha_1, \alpha_2, \ldots, \alpha_k)$, $U_2 = \mathrm{lin}(\beta_1, \beta_2, \ldots, \beta_l)$, then

$$U_1 + U_2 = \mathrm{lin}(\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_l).$$

(6) Find all subspaces of
  (a) ${\mathbb{Z}_2}^2$; (b) ${\mathbb{Z}_3}^2$; (c) ${\mathbb{Z}_2}^3$.

(7) Show that $\mathbb{R}^4 = U_1 \oplus U_2$, if
  (a) U_1 is the set of solutions of $x_1 + x_2 + x_3 + x_4 = 0$, and $U_2 = \mathrm{lin}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)$;
  (b) U_1 is the set of solutions of $\left\{ \begin{array}{c} x_1 + 2x_2 - x_3 + 3x_4 = 0 \\ -x_1 + x_2 + x_3 = 0 \end{array} \right.$, and $U_2 = \mathrm{lin}\left(\begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)$.

(8) Show that $\mathbb{R}^4 = U_1 + U_2$, but $\mathbb{R}^4 \neq U_1 \oplus U_2$, if U_1 is the set of solutions of $3x_1 - 2x_2 + x_3 + 4x_4 = 0$, and $U_2 = \operatorname{lin}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix}\right)$.

(9) Show that

$$\begin{aligned} \mathbb{R}^3 &= \operatorname{lin}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \oplus \operatorname{lin}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = lin(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\Bigg) \oplus \operatorname{lin}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) \\ &= \operatorname{lin}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \oplus \operatorname{lin}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right). \end{aligned}$$

(10) Check if the vectors α and β are linear combinations of the system $\mathcal{A}$ of vectors of the space $\mathbb{R}^4$, if

(a) $\mathcal{A} = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 2 \\ 0 \end{bmatrix}\right)$, $\alpha = \begin{bmatrix} 9 \\ 6 \\ 5 \\ -1 \end{bmatrix}$, $\beta = \begin{bmatrix} 9 \\ 6 \\ 5 \\ 0 \end{bmatrix}$;

(b) $\mathcal{A} = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}\right)$, $\alpha = \begin{bmatrix} 9 \\ 6 \\ 5 \\ -1 \end{bmatrix}$, $\beta = \begin{bmatrix} 9 \\ 6 \\ 5 \\ 0 \end{bmatrix}$.

(11) For which values of $c \in \mathbb{C}$ is the vector $\begin{bmatrix} 1 \\ i \\ i \end{bmatrix}$ a linear combination of the vectors $\begin{bmatrix} c \\ -1+i \\ 1+i \end{bmatrix}$ and $\begin{bmatrix} i \\ -1 \\ -c \end{bmatrix}$ of the space $\mathbb{C}^3$?

(12) Check if the system $\left(\begin{bmatrix} i \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ i \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ i \end{bmatrix}\right)$ of vectors of the space $\mathbb{C}^3$ is linearly independent. Express the vector $\begin{bmatrix} 2 \\ 3 \\ 1+2i \end{bmatrix}$ as the linear combination of the above vectors.

(13) Find a vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of the space $\mathbb{Z}_2^3$, such that the vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ are linearly independent. How many solutions are there?

(14) Check if the system of vectors $(\alpha_1, \dots, a_n)$ of the space K^4 is linearly independent, if

(a) $K = \mathbb{Z}_7$, $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 4 \\ 1 \\ 5 \\ 4 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 5 \\ 4 \\ 2 \\ 2 \end{bmatrix}$;

(b) $K = \mathbb{R}$, $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 4 \\ 1 \\ 5 \\ 4 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 6 \\ 3 \\ 10 \\ 5 \end{bmatrix}$;

(c) $K = \mathbb{C}$, $\alpha_1 = \begin{bmatrix} 1 \\ i \\ 3 \\ -i \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 4 \\ 1 \\ 5 \\ 4 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 4+i \\ 0 \\ 5+3i \\ 5 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 5 \\ 2i \\ i \\ 2 \end{bmatrix}$;

(d) $K = \mathbb{Z}_5$, $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 4 \\ 1 \\ 5 \\ 4 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 5 \\ 4 \\ 2 \\ 2 \end{bmatrix}$.