Problem set 6: matrices and determinants.

(1) Find the products of the following matrices:

(a) $\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} -4 & 0 \\ -1 & 5 \end{bmatrix}$, (b) $\begin{bmatrix} 6 & 4 \\ -2 & 1 \\ 7 & 9 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}$, (c) $\begin{bmatrix} -3 & 4 & 1 \\ 0 & 2 & 8 \\ 1 & 3 & -1 \end{bmatrix}^2$,

(d) $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}^3$, (e) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}^T \cdot \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$,

(f) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}^T$, (g) $\begin{bmatrix} 2 & 0 \\ 3 & 1 \\ 3 & 2 \end{bmatrix}^T \cdot \begin{bmatrix} 2 & 0 \\ 3 & 1 \\ 3 & 2 \end{bmatrix}$.

(2) For $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ find:

(a) $A^2 + 2AB + B^2$ and $(A+B)^2$;
(b) $A^2 - 2AB + B^2$ and $(A-B)^2$;
(c) $A^2 - B^2$, $(A-B)(A+B)$ and $(A+B)(A-B)$.

(3) Show that for every integer $m \geq 1$ and $a \neq 0$, $b \neq 0$, the following equalities hold true:

(a) $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^m = \begin{bmatrix} a^m & 0 \\ 0 & b^m \end{bmatrix}$, (b) $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^m = \begin{bmatrix} 1 & ma \\ 0 & 1 \end{bmatrix}$,

(c) $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}^m = \begin{bmatrix} \cos m\alpha & -\sin m\alpha \\ \sin m\alpha & \cos m\alpha \end{bmatrix}$, (d) $\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}^m = \begin{bmatrix} a^m & ma^{m-1} \\ 0 & a^m \end{bmatrix}$,

(e) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^m = \begin{bmatrix} 1 & m & \frac{m(m-1)}{2} \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix}$.

(4) Find all matrices $A \in K_2^2$, where K is an arbitrary field, such that

(a) $A\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}A$, (b) $A\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, (d) $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, (e) $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(5) Find the determinants of the following matrices:

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 1 & 4 \\ 3 & 2 & 5 \end{bmatrix}$, (b) $\begin{bmatrix} -1 & 5 & 4 \\ 3 & -2 & 0 \\ -1 & 3 & 6 \end{bmatrix}$, (c) $\begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$,

(e) $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, (f) $\begin{bmatrix} \sin\alpha & \cos\alpha & 1 \\ \sin\beta & \cos\beta & 1 \\ \sin\gamma & \cos\gamma & 1 \end{bmatrix}$ where α, β, γ are the values of the angles in a triangle,

(g) $\begin{bmatrix} 1 & \varepsilon & \varepsilon^2 \\ \varepsilon^2 & 1 & \varepsilon \\ \varepsilon & \varepsilon^2 & 1 \end{bmatrix}$, where $\varepsilon = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, (h) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & \varepsilon & \varepsilon^2 \\ 1 & \varepsilon^2 & \varepsilon^3 \end{bmatrix}$, where $\varepsilon = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$,

(i) $\begin{bmatrix} \cos\alpha\cos\beta & -r\sin\alpha\cos\beta & -r\cos\alpha\sin\beta \\ \sin\alpha\cos\beta & r\cos\alpha\cos\beta & -r\sin\alpha\sin\beta \\ \sin\beta & 0 & r\cos\beta \end{bmatrix}$.

(6) Find the following determinants:

(a) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ -3 & 2 & -5 & 13 \\ 1 & -2 & 10 & 4 \\ -2 & 9 & -8 & 25 \end{vmatrix}$, (b) $\begin{vmatrix} 1 & -1 & 1 & -2 \\ 1 & 3 & -1 & 3 \\ -1 & -1 & 4 & 3 \\ -3 & 0 & -8 & -13 \end{vmatrix}$, (c) $\begin{vmatrix} 7 & 6 & 9 & 4 & -4 \\ 1 & 0 & -2 & 6 & 6 \\ 1 & -1 & -2 & 4 & 5 \\ 1 & -1 & -2 & 4 & 4 \\ -7 & 0 & -9 & 2 & -2 \end{vmatrix}$,

(d) $\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{vmatrix}$, (e) $\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$, (f) $\begin{vmatrix} 4 & 4 & -1 & 0 & -1 & 8 \\ 2 & 3 & 7 & 5 & 2 & 3 \\ 3 & 2 & 5 & 7 & 3 & 2 \\ 1 & 2 & 2 & 1 & 1 & 2 \\ 1 & 7 & 6 & 6 & 5 & 7 \\ 2 & 1 & 1 & 2 & 2 & 1 \end{vmatrix}$,

(g) $\begin{vmatrix} 1001 & 1002 & 1003 & 1004 \\ 1002 & 1003 & 1001 & 1002 \\ 1001 & 1001 & 1001 & 999 \\ 1001 & 1000 & 998 & 999 \end{vmatrix}$, (h) $\begin{vmatrix} 30 & 20 & 15 & 12 \\ 20 & 15 & 12 & 15 \\ 15 & 12 & 15 & 20 \\ 12 & 15 & 20 & 30 \end{vmatrix}$, (i) $\begin{vmatrix} 5 & -4 & 4 & 0 & 0 & 0 \\ 9 & -7 & 6 & 0 & 0 & 0 \\ 3 & -2 & 1 & 0 & 0 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ -2 & 1 & 0 & 1 & 0 & 0 \end{vmatrix}$,

(j) $\begin{vmatrix} 1 & 6 & 20 & 50 & 140 & 140 \\ 0 & -16 & -70 & -195 & -560 & -560 \\ 0 & 26 & 125 & 366 & 1064 & 1064 \\ 0 & -31 & -154 & -460 & -1344 & -1344 \\ 0 & 4 & 20 & 60 & 176 & 175 \\ 0 & 4 & 20 & 60 & 175 & 176 \end{vmatrix}$, (k) $\begin{vmatrix} 3 & 1 & 1 & 1 & 1 & 1 \\ -1 & 3 & 1 & 1 & 1 & 1 \\ -1 & -1 & 3 & 1 & 1 & 1 \\ -1 & -1 & -1 & 3 & 1 & 1 \\ -1 & -1 & -1 & -1 & 3 & 1 \\ -1 & -1 & -1 & -1 & -1 & 3 \end{vmatrix}$.

(7) Evaluate:

(a) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 5 & 3 \\ 1 & 2 & 3 & 5 \\ 2 & 2 & 1 & 4 \end{vmatrix}$ over $\mathbb{Z}_7$, (b) $\begin{vmatrix} 1 & 1 & 1 & 2 \\ 1 & 3 & 1 & 3 \\ 1 & 1 & 4 & 3 \\ 3 & 0 & 8 & 10 \end{vmatrix}$ over $\mathbb{Z}_{11}$, (c) $\begin{vmatrix} 7 & 6 & 11 & 4 & 4 \\ 1 & 0 & 2 & 6 & 6 \\ 7 & 8 & 9 & 1 & 6 \\ 1 & 10 & 2 & 4 & 5 \\ 7 & 0 & 9 & 2 & 2 \end{vmatrix}$ over $\mathbb{Z}_{13}$.

(8) Find determinants of the following matrices of degree n:

(a) $\begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 2 \end{bmatrix}$, (c) $\begin{bmatrix} 3 & 2 & 0 & \cdots & 0 & 0 \\ 1 & 3 & 2 & \cdots & 0 & 0 \\ 0 & 1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 3 & 2 \\ 0 & 0 & 0 & \cdots & 1 & 3 \end{bmatrix}$,

(d) $\begin{bmatrix} a & 1 & 1 & 1 & \cdots & 1 \\ 1 & a & 1 & 1 & \cdots & 1 \\ 1 & 1 & a & 1 & \cdots & 1 \\ 1 & 1 & 1 & a & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & a \end{bmatrix}$, (e) $\begin{bmatrix} a & 1 & 1 & \cdots & 1 & 1 \\ -1 & a & 1 & \cdots & 1 & 1 \\ -1 & -1 & a & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & a & 1 \\ -1 & -1 & -1 & \cdots & -1 & a \end{bmatrix}$,

(f) $\begin{bmatrix} 1 & n & n & \cdots & n & n \\ n & 2 & n & \cdots & n & n \\ n & n & 3 & \cdots & n & n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n & n & n & \cdots & n-1 & n \\ n & n & n & \cdots & n & n \end{bmatrix}$, (g) $\begin{bmatrix} a & b & 0 & \cdots & 0 & 0 \\ 0 & a & b & \cdots & 0 & 0 \\ 0 & 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & a & b \\ b & 0 & 0 & 0 & 0 & a \end{bmatrix}$.

(9) Determine which of the following matrices are invertible and find their inverses where possible:

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 3 & -5 & 7 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$,

(e) $\begin{bmatrix} 2 & 3 & 2 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$.

(10) Solve the following matrix equations:

(a) $X \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 2 & 1 \end{bmatrix}$,

(b) $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} X = \begin{bmatrix} 4 & -6 \\ 2 & 1 \end{bmatrix}$,

(c) $X \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 4 & 3 & 2 \\ 1 & -2 & 5 \end{bmatrix}$,

(d) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} X \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$.

(11) Solve the following systems of matrix equations:

(a) $\begin{cases} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} X + \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} Y = \begin{bmatrix} 2 & 8 \\ 0 & 5 \end{bmatrix} \\ \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} X + \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} Y = \begin{bmatrix} 4 & 9 \\ -1 & -4 \end{bmatrix} \end{cases}$,

(b) $\begin{cases} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} X + \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} Y = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} X + \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} Y = \begin{bmatrix} 1 & 1 \\ 5 & 3 \end{bmatrix} \end{cases}$.

(12) Find inverses of the following matrices: (a) $\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$,

(c) $\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$.