

Problem set 3: complex numbers.

- (1) Find all pairs of real numbers x and y satisfying the following equations:
 - (a) $(1 + 2i)x + (3 - 5i)y = 1 - 3i$,
 - (b) $(2 + 3i)x + (4 - 5i)y = 6 - 2i$,
 - (c) $(4 - 3i)2x + (1 + i)2y = 7 - 12i$,
 - (d) $\frac{2+i}{3-i}x + \left(\frac{4-i}{3-i}\right)^2 y = 1 + i$.
- (2) Solve the following systems of equations:
 - (a) $\begin{cases} (1+i)z + (2-i)w = 2 - 2i \\ (1-i)z - (3+i)w = -3 + 3i \end{cases}$;
 - (b) $\begin{cases} (3-i)z + (4+2i)w = 2 + 6i \\ (4+2i)z - (2+3i)w = 5 + 4i \end{cases}$;
 - (c) $\begin{cases} \frac{z}{2-i} + \frac{w}{1+i} = 2 \\ \frac{z}{5z} + \frac{w}{2w} = 3 \end{cases}$.
- (3) For an arbitrary integer $n \in \mathbb{Z}$ evaluate:
 - (a) i^n ,
 - (b) $\frac{(1+i)^n}{(1-i)^{n-2}}$,
 - (c) $(1+i)^n$.
- (4) Evaluate:
 - (a) $(1+2i)^6$,
 - (b) $(2+i)^7 + (2-i)^7$,
 - (c) $(1+2i)^5 + (1-2i)^5$.
- (5) Solve the following equations:
 - (a) $z\bar{z} + (z - \bar{z}) = 3 + 2i$,
 - (b) $i(z + \bar{z}) + i(z - \bar{z}) = 2i - 3$.
- (6) What figures on the complex plane are defined by the following equations and inequalities?
 - (a) $|z| < 2$,
 - (b) $|z - 1| = 3$,
 - (c) $|z - 1 - 2i| \leq 3$,
 - (d) $1 < |z| < 5$,
 - (e) $\frac{|z-3|}{|z+1|} \geq 1$,
 - (f) $|z - c| + |z + c| = 2a$,
 - (g) $\frac{\pi}{2} < \text{Arg}(z) \leq \pi$,
 - (h) $|z - i| = |z + i|$,
 - (i) $\text{Arg} \frac{z-i}{z+i} = \frac{\pi}{2}$,
 - (j) $\text{Arg}(z - z_0) = \phi$, ϕ given,
 - (k) $0 \leq \text{Re}(iz) \leq 1$,
 - (l) $\text{Re}(z^2) > 1$.
- (7) Find the trigonometric form of the following complex numbers:

$$1, \quad -1, \quad i, \quad -i, \\ 1+i, \quad 1-i, \quad -1+i, \quad 1+i\sqrt{3}, \\ -1-i\sqrt{3}, \quad \sqrt{3}-i, \quad \sqrt{6}+\sqrt{2}+i(\sqrt{6}-\sqrt{2}), \quad \cos \frac{\pi}{3} + i \sin \frac{\pi}{6}, \\ \cos \frac{\pi}{2} + i \sin \frac{\pi}{3}.$$
- (8) Find the trigonometric form of the following complex numbers:

$$\cos \alpha - i \sin \alpha, \quad \sin \alpha + i \cos \alpha, \quad \sin \alpha - i \cos \alpha, \quad 1 + i \tan \alpha.$$
- (9) Solve the equation $z^5 = 1$.
- (10) Evaluate:
 - (a) $\frac{(1+i\sqrt{3})^{76} + 1}{(1-i)^{37}}$,
 - (b) $\frac{(1-i\sqrt{3})^{32} + 5}{(1+i)^{17}}$.
- (11) Find:

$$\sqrt{2i}, \quad \sqrt{-8i}, \quad \sqrt{3-4i}, \quad \sqrt{-15+8i}, \quad \sqrt{-3-4i}, \\ \sqrt{-11+60i}, \quad \sqrt[3]{-8i}, \quad \sqrt{-8+6i}.$$
- (12) Solve the following equations:
 - (a) $z^2 + 3z + 3 + i = 0$,
 - (b) $z^2 + (1+4i)z - (5+i) = 0$,
 - (c) $z^2 + z(1+i) + 2i = 0$,
 - (d) $(4-3i)z^2 - (2+11i)z - (5+i) = 0$.
- (13) Solve the following equations:
 - (a) $z^4 + 2z^2 + 4 = 0$,
 - (b) $z^4 + (15+7i)z^2 + 8 = 0$,
 - (c) $z^4 - (18+4i)z^2 + 77 - 36i = 0$.

(14) Solve the following equations: (a) $z^6 = \frac{(1+i)^3}{\sqrt{3}+i}$, (b) $(z+i)^n - (z-i)^n = 0$.

(15) Evaluate:

(a) $(1+2i)^6$, (b) $(2+i)^7 + (2-i)^7$, (c) $(1+2i)^5 - (1-2i)^7$.

(16) Solve the following equations:

(a) $(1+i)z^2 - (3+7i)z + 10i = 0$;

(b) $(1+2i)z^2 - (-1+8i)z + (-5+5i) = 0$;

(c) $(1+2i)z^2 - (1+7i)z + (-2+6i) = 0$;

(d) $(1+i)z^2 - (1+5i)z + (-2+6i) = 0$;

(e) $(1-i)z^2 - (7+3i)z + 10i = 0$;

(f) $(1-2i)z^2 - (4+7i)z + (7+i) = 0$;

(g) $(1+i)z^2 - (3+3i)z + (4+2i) = 0$;

(17) Evaluate

(a) $\frac{(1-i)^{24}}{(\sqrt{3}-i)^{22}}$; (b) $\frac{(1-i\sqrt{3})^{42}}{(-1+i)^{31}}$; (c) $\frac{(-1+i\sqrt{3})^{36}}{(1+i)^{31}}$; (d) $\frac{(1-i)^{28}}{(\sqrt{3}+i)^{20}}$;

(e) $\frac{(1-i)^{28}}{(\sqrt{3}+i)^{20}}$; (f) $\frac{(-1+i)^{32}}{(-\sqrt{3}+i)^{28}}$; (g) $\frac{(-1-i)^{28}}{(1-i\sqrt{3})^{20}}$.