Problem set 2: groups, rings and fields.

(1) (a) Find all possible values of the function $x \mapsto x^2$ in $\mathbb{Z}_{11}$.
(b) Find all possible values of the function $x \mapsto x^{-1}$ in $\mathbb{Z}_{13}$.
(c) Find the domain and all possible values of the function $x \mapsto \dfrac{x+2}{2x-1}$ in $\mathbb{Z}_7$.

(2) Solve the following equations:
(a) $5x^2 + 5x + 1 = 0$ in $\mathbb{Z}_{11}$, (b) $x^2 + x + 3 = 0$ in $\mathbb{Z}_5$,
(c) $2x^2 + 2x + 2 = 0$ in $\mathbb{Z}_{13}$, (d) $2x^3 + 3x^2 + x - 4 = 0$ in $\mathbb{Z}_7$.

(3) For which values of $m \in K$ the equation $mx^2 + (2m+1)x + m - 2 = 0$ has two different solutions in K
(a) if $K = \mathbb{Z}_{11}$? (b) if $K = \mathbb{Z}_{13}$?

(4) Show that every element of $\mathbb{Z}_5$ is a cube. Same for $\mathbb{Z}_{11}$. How about $\mathbb{Z}_{13}$?

(5) Check if there exist (and, if so, find them) square roots of -1 in $\mathbb{Z}_p$ for $p = 2, 3, 5, 7, 11, 13$.

(6) Solve the equation $5x = 2$ in $\mathbb{Z}_{65537}$.

(7) Find an element of $\mathbb{Z}_5$ such that every element other than 0 is a power of this element. Same for $\mathbb{Z}_7$ and $\mathbb{Z}_{11}$.

(8) Check that every element of $\mathbb{Z}_5$ other than 0 raised to a certain nonzero power gives 1. Same for $\mathbb{Z}_7$ and $\mathbb{Z}_{11}$.

(9) Evaluate $(6^2 \cdot 3 + 5 \cdot 4 - 1) \cdot (5 \cdot 12 - 7)^{-1}$ in $\mathbb{Z}_{17}$ and $\mathbb{Z}_{23}$.

(10) Solve the following systems of equations
(a) $\begin{cases} 3x + 5y = 2 \\ 4x + 9y = 4 \end{cases}$ in $\mathbb{Z}_{13}$ and $\mathbb{Z}_7$ (b) $\begin{cases} 5x + 4y = a \\ 4x + 3y = b \end{cases}$ in $\mathbb{Z}_{11}$ and $\mathbb{Z}_5$.

(11) Find the number of solutions of the system $\begin{cases} 3x + 4y = 2 \\ 9x + y = 7 \end{cases}$ in $\mathbb{Z}_{11}$. Same for $\mathbb{Z}_{13}$ and $\mathbb{Z}_{17}$.