ELEMENTS OF LOGIC AND SET THEORY

(*Elementy logiki i teorii zbiorów*)

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1 Basic notions

We assume that the notion of **set** is a primitive notion, hence it is not defined. However, we can define and describe particular sets, namely by listing or describing their elements:

$\mathbb{Q} = \{x \in \mathbb{R};$ such that there are $k \in \mathbb{Z}, l \in \mathbb{Z}$ such that $l \neq 0$ and $x = k/l\}$

This is $\mathbb{Q}$ - the set of rational numbers. The empty set is denoted by $\emptyset$ (there is only one empty set!).

The notation $x \in B$ should be read as "x belongs to B", while $A \subseteq B$ means "A is *contained* in B", i.e. if $x \in A$, then $x \in B$. In such a case we say that B is a *superset* of A and A is a *subset* of B.

Powerset of A is $P(A)$ and it is a collection of all subsets of A, i.e. $B \in P(A) \Leftrightarrow B \subseteq A$. Note that for any A, $\emptyset \subseteq A$. Hence, $P(\{1\}) = \{\emptyset, \{1\}\}$, $P(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$.

Basic **operations** are:

1. Union (of sets): $\cup$, $x \in A \cup B \Leftrightarrow x \in A$ **or** $x \in B$.

2. Intersection: $\cap$, $x \in A \cap B \Leftrightarrow x \in A$ **and** $x \in B$.

3. Difference: $\setminus$, $x \in A \setminus B \Leftrightarrow x \in A$ and **not in** B, i. e. $A \setminus B = \{x; x \in A$ and $x \notin B\}$

4. Symmetric difference: $\ominus$, $x \in A \ominus B \Leftrightarrow (x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A)$, i.e. $A \ominus B = (A \cup B) \setminus (B \cap A) = (A \setminus B) \cup (B \setminus A)$.

5. If $\mathcal{A}$ is a family of sets, then $\bigcup \mathcal{A}$ is its union, i.e. if $x \in \bigcup \mathcal{A}$, then there is $A \in \mathcal{A}$ such that $A \in \mathcal{A}$.

Examples

1. If $A = \{A \in \mathbb{R}; f(x) = 0\}, B = \{x \in \mathbb{R}; g(x) = 0\}$, then $A \cup B = \{x \in \mathbb{R}; f(x) \cdot g(x) = 0\}$

2. If $\mathcal{A} = \{A \in P(\mathbb{R})$ such that there is $n \in \mathbb{N}, n \neq 0, A = [\frac{1}{n}; 2 - \frac{1}{n}]\}$, then $\bigcup \mathcal{A} = (0, 2)$.

3. If $A = \{x \in \mathbb{R}; f(x) = 0\}, B = \{x \in \mathbb{R}; g(x) = 0\}$, then $A \cap B = \{x \in \mathbb{R}; f^2(x) + g^2(x) = 0\}$.

4. If $\mathcal{A} = \{A \in P(\mathbb{R})$ such that there is $n \in \mathbb{N}, n \neq 0, A = (0, 1/n)\}$, then $\bigcap \mathcal{A} = \emptyset$.

De Morgan laws:

$A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$ $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Cartesian product of two sets: $A \times B = \{\langle a, b \rangle; a \in A, b \in B\}$.

Intuitively, **cardinality** of the set A is its "number of elements". It is denoted by $|A|$, e.g. $|\{1,2,3,4,5\}| = 5$. However, this number can be infinite. In fact, we have various kinds of infinity: the number of elements of $\mathbb{N}$ is usually denoted by $\aleph_0$; cardinality of $\mathbb{R} = \mathfrak{c}$. It can be shown that $|\mathbb{N}| < |\mathbb{R}|$, $|\mathbb{R}| = |P(\mathbb{N})|$, $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$, $|\mathbb{N}| = |\mathbb{Q}|$ or $|\mathbb{Z}| = |\mathbb{N}|$.

2 Functions and relations

Function is a set which elements are ordained pair of the form (x, y) and: **1**) for any $z \in f$ there are x, y such that $z = \langle x, y\rangle$; **2**) for any x, y_1, y_2: if $\langle x, y_1\rangle \in f$ and $\langle x, y_2\rangle \in f$, then $y_1 = y_2$.

Examples:

1. $f = \{\langle x, y\rangle \in \mathbb{R} \times \mathbb{R}; y = x^2\}$
2. $g = \{\langle 1, 1\rangle, \langle 2, 2\rangle, \langle 3, 1\rangle, \langle 4, 2\rangle, \langle 5, 3\rangle\}$. We see that $g(1) = 1, g(2) = 2, g(3) = 1, g(4) = 2, g(5) = 3$.

3 Relations

Any $R \subseteq X_1 \times X_2 \times ... \times X_n$ is called n-ary relation. Assume that we have only X_1 and X_2. Moreover, $X_1 = X_2 = X$, i.e. we are working with $X \times X$. Then $R \subseteq X \times X$ is called *binary relation*. Binary relations can be (of course these are only selected possible properties):

1. Reflexive: for any $x \in X$, xRx.
2. Symmetric: for any $x_1, x_2 \in X$, $x_1Rx_2 \Rightarrow x_2Rx_1$.
3. Transitive: for any $x_1, x_2, x_3 \in X$, if x_1Rx_2 and x_2Rx_3, then x_1Rx_3.
4. Anti-symmetrical: for any $x_1, x_2 \in X$, if $(x_1Rx_2)and(x_2Rx_1)$, then $x_1 = x_2$.
5. Connex: for any $x_1, x_2 \in X$ we have x_1Rx_2 or x_2RX_1.

If our relation holds properties:

- $1, 2, 3$, then we speak about *equivalence relation*.
- $1, 3, 4$, then (...) *partial order*.
- $1, 3$, then (...) *preorder* (or *quasi-order*).
- $1, 3, 4, 5$, then (...) *linear oder*.

In case of the equivalence relation we can speak about classes of equivalence: $R(x) = \{y \in X; xRy\}$. If $R(x) \neq R(y)$, then $R(x) \cap R(y) = \emptyset$.

Examples:

1. $x_1|x_2 \Leftrightarrow$ there is $n \in \mathbb{N}$ such that $nx_1 = x_2$. This is partial order.
2. $x_1\alpha x_2 \Leftrightarrow |x_1| \leqslant |x_2|$. This is preorder (but not partial order).
3. Let R be relation on the set $U = \mathbb{Z}\times\mathbb{Z}_1$, where $\mathbb{Z}_1 = \mathbb{Z}\setminus\{0\}$. Assume that $(m, n)R(p, q) \Leftrightarrow mq = np$. This is equivalence and rational numbers are its classes of equivalence (for example, $3/4, 6/8$ and $12/16$ belong to the R($3/r$), they are equivalent with $3/4$).
4. Let $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$ and $x_1Rx_2 \Leftrightarrow \frac{x_1}{x_2} > 0$. There are two equivalence classes: $\mathbb{Q}^*_{|R} = \{R(\text{-}1), R(1)\}$.

4 Some logic

Let us discuss the set L of all theorems which can be inferred by *modus ponens* (i.e. $\varphi, \varphi \to \gamma \vdash \gamma$) from the set:

$\varphi \to (\gamma \to \varphi)$

$(\varphi \to (\gamma \to \psi)) \to ((\varphi \to \gamma) \to (\varphi \to \psi))$,

$(\neg\gamma \to \neg\varphi) \to (\varphi \to \gamma)$

It can be shown (by means of so-called *soundess and completeness* theorem) that L is equal with the set of all formulas which have value 1 (or T) in the typical truth-table.

Examples:

p	q	$q \to p$	$p \to (q \to p)$
0	0	1	1
0	1	0	1
1	0	1	1
1	1	1	1

p	q	$p \vee q$	$p \to (p \vee q)$
0	0	0	1
0	1	1	1
1	0	1	1
1	1	1	1

Here we have truth-tables for classical connectives:

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

$\to$	0	1
0	1	1
1	0	1

(classical implication if false only if we have $1 \to 0$)