Assignment 1

due date: Monday, July 5th, 2010.

The following is a list of some drill questions. The five problems marked with **(*)** is your assigned homework.

Sums of powers of consecutive integers.

1. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2} = s_1$
2. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = s_2$ (see above)
3. $\sum_{k=1}^{n} k^3 = s_1^2$
4. $\sum_{k=1}^{n} k^4 = \frac{1}{5}s_2(3n^2 + 3n - 1)$
5. $\sum_{k=1}^{n} k^5 = \frac{1}{3}s_1^2(2n^2 + 2n - 1)$
6. **(*)** $\sum_{k=1}^{n} k^6 = \frac{1}{7}s_2(3n^4 + 6n^3 - 3n + 1)$
7. $\sum_{k=1}^{n} k^7 = \frac{1}{6}s_1^2(3n^4 + 6n^3 - n^2 - 4n + 2)$
8. $\sum_{k=1}^{n} k^8 = \frac{1}{15}s_2(5n^6 + 15n^5 + 5n^4 - 15n^3 - n^2 + 9n - 3)$

Sums of powers of consecutive odd integers.

1. $\sum_{k=1}^{n}(2k-1) = n^2 = \sigma_1$
2. $\sum_{k=1}^{n}(2k-1)^2 = \frac{1}{3}n(4n^2 - 1) = \sigma_2$
3. $\sum_{k=1}^{n}(2k-1)^3 = \sigma_1(2n^2 - 1)$
4. $\sum_{k=1}^{n}(2k-1)^4 = \frac{1}{5}\sigma_2(12n^2 - 7)$
5. **(*)** $\sum_{k=1}^{n}(2k-1)^5 = \frac{1}{3}\sigma_1(16n^4 - 20n^2 + 7)$
6. $\sum_{k=1}^{n}(2k-1)^6 = \frac{1}{7}\sigma_2(48n^4 - 72n^2 + 31)$

Divisibility.

1. $2|n^2 - n$
2. $6|n^3 - n$
3. $30|n^5 - n$
4. $42|n^7 - n$
5. $546|n^13 - n$
6. $9|10^n - 1$
7. $12|10^n - 4$
8. $11|10^n - (-1)^n$

9. $101|10^{2n}-(-1)^n$
10. **(*)** $1001|10^{3n}-(-1)^n$
11. $7|10^{3n+1}-3(-1)^n$
12. $13|10^{3n+1}+3(-1)^n$
13. $14|10^{3n+2}-2(-1)^n$
14. $52|10^{3n+2}+4(-1)^n$
15. $11|2^{6n+1}+3^{2n+2}$
16. $10|2^{2^n}-6$
17. $41|5\cdot 7^{2(n+1)}+2^{3n}$
18. $25|2^{n+2}3^n+5n-4$
19. $169|3^{3n}-26n-1$
20. $11|5^{5n+1}+4^{5n+2}+3^{5n}$

Bernoulli inequality and some of its generalizations.

1. $(1+a)^n \geq 1+na$, $a>-1$ (Bernoulli, 1689)
2. $(1+a)^n \geq 1+na+\frac{n(n-1)}{2}a^2$, $a \geq 0$
3. $(1+a)^n \geq 1+na+\frac{n(n-1)}{2}a^2+\frac{n(n-1)(n-2)}{6}a^3$, $a>-1$
4. $(1+a)^{1/n} \leq 1+\frac{a}{n}$, $a>-1$
5. $(1+a)^{1+1/n} \geq 1+\frac{a}{n(1+a)}$, $a>-1$
6. **(*)** $(1+a)^{1+1/n} \geq 1+(1+\frac{1}{n})a$, $a>-1$
7. $(1+a)^{1+m/n} \geq 1+(1+\frac{m}{n})a$, $a>-1$
8. $(1+a)^{p/q} \geq 1+\frac{p}{q}a$, $a>-1$, $p \geq q \geq 1$
9. $(1+a)^{p/q} \leq 1+\frac{p}{q}a$, $a>-1$, $1 \leq p \leq q$

Fibonacci sequence. Set

$$u_0=0, \qquad u_1=1,$$

$$u_{n+2}=u_n+u_{n+1}$$

1. $\sum_{k=0}^{n} u_k = u_{n+2}-1$
2. $\sum_{k=0}^{n} u_{2k+1} = u_{2n+2}$

3. $\sum_{k=0}^{n} u_{2k} = u_{2n+1} - 1$
4. $\sum_{k=0}^{2n} (-1)^k u_k = u_{2n-1} - 1$
5. $\sum_{k=0}^{2n+1} (-1)^{k+1} u_k = u_{2n} + 1$
6. $\sum_{k=0}^{n} u_k^2 = u_n u_{n+1}$
7. $\sum_{k=0}^{2n-1} u_k u_{k+1} = u_{2n}^2$
8. (*) $u_{n-1}u_{n+1} - u_n^2 = (-1)^n$
9. $u_{n+1} = u_m u_{n-m} + u_{m+1} u_{n-m+1}$, $n \geq m \geq 0$
10. $u_{2n+1} = u_n^2 + u_{n+1}^2$
11. $u_{2n} = u_{n+1}^2 - u_{n-1}^2$
12. $u_{3n} = u_n^3 + u_{n+1}^3 - u_{n-1}^3$
13. $u_n^4 = 1 + u_{n-2}u_{n-1}u_{n+1}u_{n+2}$
14. $\sum_{k=0}^{n} \frac{u_{k+2}}{u_{k+1}u_{k+3}} = \frac{u_3}{u_1 u_2} - \frac{u_{n+4}}{u_{n+2}u_{n+3}}$
15. $u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, where α i β are two distinct solutions of the equation

$$x^2 = x + 1.$$